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6 A MIXED-INTEGER PROGRAMMING APPROXIMATION TO THE STOCHASTIC MULTISTAGE INVENTORY MODEL.

10 Donald R. Edwards, Lt Col, USAF

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The study was concerned with the essential question of how to address multiperiod inventory problems characterized by not unrealistic conditions for which modeling and solution procedures have not been developed. The research placed special emphasis on the application of deterministic mixed-integer programming models to multiperiod inventory problems characterized by changing costs and beta-distributed demands. The special concern for the mixed-integer programming model was prompted by the realization that, among all of the easy-to-use deterministic inventory models, the mixed-integer programming formulation is the only model that is amenable to the additional constraints and multiple-objective criteria that coincide with broadly conceived statements of inventory control. Through the combined usage of mixed-integer programming and computer simulation techniques, a method was developed whereby a first-period reorder policy that minimizes expected total inventory cost over a multiperiod planning horizon can be identified with a nominal investment in computer processing time. The analysis led to the conclusion that first-period policies obtained by using the mixed-integer programming model with expectations as periodic demand inputs are generally adequate under the conditions specified in the research and compare favorably with policies obtained from commonly used inventory models.

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A School of Systems and Logistics AU-AFIT-LS Technical Report

Air University

Air Force Institute of Technology

Wright-Patterson AFB, Ohio

By

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Lieutenant Colonel, USAF

May 1978

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CHAPTER I

INTRODUCTION

Statement of the Problem

This research is concerned with the identification of inventory control methods for solving multiperiod inventory problems having the following characteristics:

1. Inventory costs and inventory demands can change from period to period.

2. Future period demands are subject to uncertainty. Probability assessments of future demands are based on potentially limited information; for example, information that is analogous to the three-parameter assessments assumed in the Program Evaluation and Review Technique (PERT) for characterizing uncertain activity times.

3. Inventory control is potentially subject to a variety of restrictions related to organizational requirements. This condition renders desirable constrained optimization procedures for inventory management.

4. The inventory control problem can be viewed as a multiple-objective problem. For example, the minimization of opportunity costs and the maximization of turnover might be equally compelling objectives.

5. The decision maker is able to modify inventory policies from period to period as previously random events are realized.

6. The decision maker's limited analytical abilities restrict him to use of well-known inventory models that impose minimal mathematical and computational burdens, or to easily formulated models that can be solved by computers using readily available optimization packages.

In the absence of condition 2 (uncertainty concerning future demands), the conditions postulated above could be satisfied by recourse to a deterministic, mixed-interger programming formulation of the inventory problem. The ability to invoke goal programming methods within the format of integer programming renders the approach appropriate to multi-objective, constrained, multiperiod inventory control. Furthermore, in the absence of uncertainty, there is no need for policy revision.

In the absence of conditions 1, 2, 3, and 4 (changing costs or demands, uncertainty, constraints, and multiple objectives), the inventory control problem could be solved by recourse to well-known deterministic inventory models such as the EOQ model, the (t_p,S) model, or the (s,S) model.

In the absence of condition 6 (limited analytical ability), the inventory problem might be solved by recourse to dynamic programming methods. At this time, only a very few special cases of stochastic, multiperiod, constrained inventory models have been solved, and at considerable analytical and computational effort.

As such, this research addresses a gray area that falls beyond the established methods of management science. The essential question concerns how to address inventory control when faced with not unrealistic conditions for which modeling and solution procedures have not been developed.

The pragmatic strategies that are considered in this research all entail the use of deterministic inventory models. In all cases, random variables are replaced by deterministic proxies--for example, future random demands are replaced with expectations--to provide (nonoptimal) policies for implementation over a multiperiod planning horizon. Such policies are implemented only during the immediate period. At the end of the period, expectations and policies are revised based on conditions that were realized during the period.

The problem addressed by this research concerns not only the evaluation of the relative performance of several pragmatic strategies, but also places special emphasis on determining the best means of using deterministic mixed-integer programming models as proxies for stochastic multi-period inventory models. The special concern for the mixed-integer programming model is based on the realization that, among all of the easy-to-use deterministic inventory models, the mixed IP formulation is the only model that is amenable to the additional constraints and multiple-objective

criteria that coincide with broadly conceived statements of inventory control.

Objectives of the Study

A major objective of the study is to determine whether total cost over a four-period planning horizon is sensitive to the reorder policy that is implemented for the first period. To answer this question, it is necessary to compute expected total costs using various reorder policies for period 1 and optimal policies (contingent on the first-period policy) for the remaining three periods. Such computations, for a broad sample of cost and beta-distributed demand patterns, permit plotting expected total cost over the planning horizon as a function of the quantity specified by the first-period reorder policy. If the resulting curve indicates that there is no unique minimum expected total cost, then concern for determining an optimal policy for any one period vanishes. This realization leads to a statement of the first hypothesis of this study.

Hypothesis 1: Minimum ETC $[X_1 | X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is not unique; where ETC $[X_1 | X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is the expected total cost over a four-period planning horizon, given an inventory policy, X_1, for the first period and optimal policies for the remaining three periods. Rejection of Hypothesis 1 implies the existence of an optimal (and identifiable) first-period policy.

A second major objective of the research is to ascertain whether an optimal first-period reorder policy can be easily and economically determined for a four-period, finite horizon inventory problem characterized by changing costs and demands. This objective is pursued by addressing the second hypothesis of the study.

Hypothesis 2: By using a combination of simulation and mixed-integer programming methods, an optimal first-period policy, X_1, can be determined over a four-year planning horizon with a nominal investment in computer processing time.

A third objective of the study is to ascertain the magnitude of error that is introduced by applying a deterministic proxy--specifically, a mixed-integer programming formulation--for a probabilistic inventory problem. In order to pursue this objective, it is necessary to assess the adequacy of first-period reorder policies obtained by using the mixed IP model when expectations are used as demand inputs. This goal is accomplished by addressing the third hypothesis.

Hypothesis 3: $ETC(X_1{*}) = ETC(X_1^{opt})$; where $ETC(X_1{*})$ is the expected total cost over a four-period planning horizon resulting from the implementation of first-period reorder policies derived from the mixed IP formulation. $ETC(X_1^{opt})$ is the expected total cost incurred as a result of implementing optimal first-period policies. Acceptance of

Hypothesis 3 implies that the difference between $ETC(X_1*)$ and $ETC(X_1^{opt})$ cannot be shown to be statistically significant.

The fourth research objective is concerned with evaluating the performance of the mixed-integer programming formulation relative to the performances of the EOQ model, the (t_p,S) model, and the deterministic (s,S) model, when each is employed in the solution of a multiperiod problem characterized by beta-distributed demands. To accomplish this objective, an experiment is designed in which each model is used to solve a sample of inventory problems in which reorder decisions must be made during each of twenty consecutive periods. Each reorder decision is based on available cost and demand estimates over a four-period finite planning horizon.

Tests of the following null hypotheses are conducted in order to assess the comparable performances of the models.

<u>Hypothesis 4</u>: $ETC(IP) \geq ETC(EOQ1)$; where ETC(IP) is the expected total cost (over twenty consecutive periods) resulting from the implementation of a mixed-integer programming model. ETC(EOQ1) is the expected total cost when the EOQ model is employed by using estimates of the most likely next-period demands as proxies for the constant demands assumed by the model.

<u>Hypothesis 5</u>: $ETC(IP \geq ETC(EOQ2)$; where ETC(EOQ2) is the expected total cost resulting from employing the EOQ model when demand inputs are determined by averaging demand estimates over the next four periods.

<u>Hypothesis 6</u>: $ETC(IP) \geq ETC(t_p,S1)$; where $ETC(t_p,S1)$ is the expected total cost given the implementation of the (t_p,S) model when next-period estimates are used as demand inputs.

<u>Hypothesis 7</u>: $ETC(IP) \geq ETC(t_p,S2)$; where $ETC(t_p,S2)$ is the expected total cost resulting from employing the (t_p,S) model when four-period averages are used as demand inputs.

<u>Hypothesis 8</u>: $ETC(IP) \geq ETC(s,S1)$; where $ETC(s,S1)$ is the expected total cost resulting from using the deterministic (s,S) model when demand inputs are based on estimates of most likely next-period demands.

<u>Hypothesis 9</u>: $ETC(IP) \geq ETC(s,S2)$; where $ETC(s,S2)$ is the expected total cost that results when the (s,S) model is implemented by using four-period averages as demand inputs.

Rejection of any of these nine hypotheses implies that, given the conditions set forth in the research, the mixed-integer programming model performs at least as well as the model specified by the hypothesis.

The study is further concerned with determining whether the performance of the mixed-integer programming model, relative to that of each of the other three models, is sensitive to the ratio of stockout cost to holding cost. This objective is pursued by conducting six linear regressions to ascertain whether the difference between ETC(IP) and the expected total cost resulting from employing each

version of the other three models is significantly dependent upon the value of the cost ratio.

Significance of the Study

It is believed that certain multiperiod inventory problems are more representative of the environment in which much inventory control is practiced than are the conditions postulated by common inventory models. It is not uncommon, for example, to encounter multiperiod inventory problems in which (a) costs and demands vary from period to period, (b) future periodic demands are characterized by uncertainty, (c) the practitioner must take a global view of inventory in order to be responsive to externally imposed constraints and/or multiple objectives, (d) policies may be modified from period to period as previously random events are realized, and (e) the decision maker's analytical capabilities are restricted to the use of computationally simple models and/or the use of readily available computer programs for optimization. This research provides results to guide practitioners in selecting an appropriate solution procedure, given the postulated conditions.

To the extent that this research explores an important yet little studied area, it provides specific findings of interest to the practitioner. It also provides a procedure that can be followed by other researchers in further explorations of how to proceed in the gray area that exists

between the use of well-known inventory models and the conditions in which they must necessarily be implemented.

The ability to efficiently solve problems cast as mixed IP models is a comparatively new development. It has been recognized for some time that the ability to solve such models should allow practitioners to take a more globally conceived approach to inventory problems, but relatively little computational and implementation experience is available. The findings of this research serve to demonstrate the feasibility and desirability of this type of approach.

One of the significant developments in inventory control during the past decade has been the advent of material requirements planning (MRP) systems. MRP provides the data base and data-management systems that are essential for the implementation of improved inventory control procedures. The coupling of MRP with powerful optimization procedures provides an opportunity for enhancing and extending the ability of practitioners to deal with real-world inventory problems.

Scope and Limitations of the Study

The research is limited to the case in which a decision maker has available three-parameter estimates for demands in each of four successive periods. The three parameters correspond to estimates of (a) the least possible demand, (b) the most likely (modal) demand, and (c) the greatest possible demand. It may be noted that these

three-parameter estimates coincide with the activity time estimates that have achieved widespread application in the program evaluation and review technique (PERT). The estimates further coincide with the types of assessments that managers could reasonably be expected to develop when future demands are estimated (at least in part) using naive forecasting methods such as exponential smoothing or seasonally adjusted moving averages.

For the purposes of the research, it is assumed that future demands may be further characterized by a beta-distributed random variable. For each three-parameter demand estimate, there exists a family of (two-parameter) beta density functions. To identify a specific beta density, it is necessary to make an assumption concerning its variance. In this regard, it is assumed that the decision maker desires to establish inventory policies that are consistent with the typical PERT assumption that the standard deviation of period demand is approximately equivalent to one-sixth of the domain of definition; that is, one-sixth of the difference between the greatest possible and the least possible demands.

The assumption that period demands may be reasonably approximated by a beta-distributed random variable is prompted by the richness of the beta, the ease with which distribution parameters may be estimated, and the considerable success that has been attained in its application in a similar fashion

in PERT. MacCrimmon and Ryavec [1964] have suggested that the triangular distribution would have been preferable to the beta in the PERT analysis. This suggestion was based on the fact that the use of the triangular distribution would introduce no error due to using estiamted parameters since its mean and standard deviation may be calculated exactly, given the three-parameter estimate. The decision to use the beta in this study, in view of this argument, was influenced by the relatively small errors (except in extreme cases) caused by parameter estimation,[1] and the widespread familiarity of the beta to practitioners.

The mixed-integer programming formulation is limited to consideration of the "lost sales" case in which backlogged orders are not filled. The model further assumes (a) constant <u>intra</u>period demand and withdrawal rates, (b) instantaneous replenishment at the beginning of any period in which a reorder is made, (c) one product, and (d) a single vendor. Assumptions (b), (c), and (d) may be easily relaxed by the introduction of additional constraints and variables. Assumption (a) is essential; however, the ability to partition a planning horizon into n (finitely many) subperiods permits

[1]MacCrimmon and Ryavec showed that the possible errors resulting from the assumption that $\mu = 1/6(b-a)$ and the approximation $\sigma = 1/6(a+4m+b)$ could be quite large (up to 33 percent for the mean and 17 percent for the standard deviation) for extreme parameter values. When the parameters are restricted to more reasonable ranges, however, the errors in the mean and standard deviation reduce to 4 percent and 7 percent, respectively.

the use of piecewise linear approximations to nonlinear demand patterns.

The population of periodic demand distributions used in the study is restricted to those distributions that have skewness parameters (defined as the ratio of the right semi-range to the range)[2] within the interval 0.1 to 0.9, and that have modal values that fall within the range 0 to 100 units per period. Thus the smallest demand per period that could ever occur is 0, and the largest that could ever occur would be 1,000. The restrictions on the modal and skewness parameters were imposed to preclude unreasonable variability in demand patterns from period to period, while still allowing for the consideration of a large variety of demand distributions.

The population of holding costs used in the research is limited to values falling within an interval corresponding to 20 to 40 percent of the cost value of the stocked item. This range is consistent with the trend reported in studies by Whitin [1966] and Nelleman and Thiry [1970]. Stockout costs, which are considerably more difficult to determine in practice, are restricted to a range corresponding to 30 to 60 percent of the cost value of the stocked item. Reorder costs are assumed to take on, with equal

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[2]This measure of skewness was chosen, in preference to the more commonly used coefficient of skewness, because its linearity permits using uniform random fraction generators to generate random demand distributions.

probabilities, values corresponding to two, four, six, eight, and ten times the cost value of the item of inventory.

The evaluation of comparative model performance is limited to consider, in addition to the mixed-integer programming model, three well-known deterministic inventory models--specifically, the EOQ model, the (t_p,S) model, and the (s,S) model. This choice is based on the realization that these models are characterized by a degree of computational tractability that is consistent with the abilities of most practitioners.

Organization of the Study

Five chapters follow this introductory chapter. Chapter II describes the methodology followed in the course of the study. The chapter discusses the generation and validation of sample data, as well as the analytical and statistical techniques that were employed in subsequent analyses.

Chapter III is concerned with the statement of models evaluated in this research, as well as the formal specification of how the models can be implemented over time when the decision maker is able to revise inventory policies based on current realizations.

Chapter IV presents the results of tests of hypotheses concerning (a) the sensitivity of expected total cost over a four-period planning horizon to first-period reorder policies, (b) the feasibility of economically determining optimal first-period reorder policies, and (c) the adequacy

of first-period reorder policies derived from using the mixed-integer programming model with expectations as demand inputs.

The fifth chapter presents the results of statistical tests conducted to compare the performance of the mixed-integer programming model with those of the EOQ, (t_p,S), and (s,S) models when each is implemented under the conditions postulated by the research methodology.

The final chapter, Chapter IV, contains a summary, conclusions, and recommendations for additional research in the area.

CHAPTER II

METHODOLOGY

The methodology followed in this research involves extensive data generation and the development of a procedure whereby expected total cost over a four-period planning horizon can be depicted as a function of the first-period reorder policy. The methodology also calls for an analysis of the adequacy of mixed-integer programming solutions to the multi-period inventory problem characterized by beta-distributed demands, when expectations are used as demand inputs. Finally, the methodology involves an analysis in which the performance of the mixed-integer programming inventory model is compared to the performances of three well-known deterministic inventory models.

Data Generation

The research is limited to the multiperiod case in which periodic demands are assumed to be distributed according to a generalized beta distribution (not restricted to the 0-1 interval). Periodic demands are generated via a linear transformation of the beta distribution. The beta is a very rich distribution that is completely specified over a 0-1 range by two parameters, α and β. The form of the beta used in this study is the familiar one in which the

mean, μ, and standard deviaion, σ, are assumed to be determined by

$$\mu = (A+4M+B)/6 \qquad (3.1)$$

$$\sigma = (B-A)/6 \qquad (3.2)$$

where A represents the least possible value of the random variable, B is the greatest possible value, and M is the mode.

The planned methodology necessitates the generation of 675 generalized beta distributions. The generation of a distribution is initiated by using a uniform random fraction generator to obtain the skewness measure, $K = (B-M)/(B-A)$, where K is restricted to values between 0.1 and 0.9. This range is believed to encompass those demand distributions that might reasonably be expected to occur.

The modal demand, M, is next generated over the range from 0 to 100, using a uniform fraction generator. The least possible demand, A, where $0 \leq A \leq M$, is then generated in the same manner. The greatest possible demand, B, is next computed by solving $B = (M-AK)/(1-k)$.

Beta-distributed demands. To generate a random sample of demands from the specified distributions, it is first necessary to solve for μ and σ by using relationships (3.1) and (3.2). By transforming these values to a 0-1 interval, α and β can be determined by solving simultaneously

$$\mu = \alpha/(\alpha+\beta) \qquad (3.3)$$

and

$$\sigma = \sqrt{(\alpha\beta)/(\alpha+\beta)^2\ (\alpha+\beta+1)}. \qquad (3.4)$$

This yields

$$\alpha = 36\left[\mu^2(1-\mu) - \mu/36\right] \qquad (3.5)$$

and

$$\beta = \alpha(1-\mu)/\mu. \qquad (3.6)$$

α and β are then provided as inputs to a computer code in order to generate a beta-distributed random fraction. A linear transformation of this value is accomplished by multiplying it by (B-A) and adding the result to A to obtain a demand that follows a generalized beta distribution. The research methodology calls for the generation of 30 such demands from each randomly generated distribution.

Validation of the beta random number generator was accomplished by subjecting 10 randomly selected distributions to goodness-of-fit tests. One hundred demands were generated from each distribution tested. The computer code used to conduct the tests employed the Kolomogorov-Smirnov test at the .05 significance level. All of the 10 distributions tested were accepted as being beta distributions. A summary of the results of the goodness-of-fit tests is presented in Table 1.

<u>Costs of inventory</u>. The methodology of the study requires the generation of a random sample of 125 sets of

Table 1

Results of Goodness-of-Fit Tests

Distribution Number	Kolomogorov-Smirnov Test Statistic[a]	Maximum Absolute Error[b]
1	.1360	.0536
2	.1360	.0607
3	.1360	.0465
4	.1360	.0556
5	.1360	.0503
6	.1360	.1136
7	.1360	.0937
8	.1360	.038[illegible]
9	.1360	.0472
10	.1360	.0517

[a]Based on sample size, n, = 100, at .05 significance level.

[b]The maximum allowable error is the maximum observed difference between actual and theoretical cumulative cell probabilities. If the error is less than the test statistic, the distribution is accepted.

inventory costs. Each set consists of a holding cost (\$/unit/period), a stockout cost (\$/unit short), and a reorder cost (\$/reorder). Each periodic holding cost is determined by using a uniform random fraction generator to select a value over the range \$2 to \$4 (corresponding to 20 to 40 percent of the item cost value, which is assumed to be \$10).[4] The unit stockout cost is generated in a similar manner to find a value between \$3 and #6. Each periodic reorder cost is computed by using a uniform random fraction generator to choose among five possible values (\$20, \$40, \$60, \$80, and \$100) that occur with equal probability.

The Development of Expected Total Cost Functions

Research hypotheses 1 and 2 are restated for the convenience of the reader.

Hypothesis 1: Minimum ETC $[X_1 \mid X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is not unique; where ETC $[X_1 \mid X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is the expected total cost over a four-period planning horizon, given an inventory policy, X_1, for the first period and optimal policies for the remaining three periods.

Hypothesis 2: When a combination of simulation and mixed-integer programming methods is used, an optimal first-period policy, X_1, can be determined over a four-period

[4]Note that the item cost value is merely an artifact whose absolute value is not important. It is used to obtain holding costs, stockout costs, and reorder costs in relative proportions that are likely to be encountered in practice.

planning horizon with a nominal investment in computer processing time.

To test these two hypotheses, 100 of the randomly generated demand distributions and 100 of the randomly generated sets of inventory costs are used to describe 25 simulated four-period planning horizons. The beginning inventory level for each planning horizon is determined by generating a uniform random fraction and multiplying this value by the first demand generated during the first period. The periodic inventory costs, the beginning inventory levels, and the randomly generated beta-distributed demands provide the input parameters necessary to formulate 750 four-period inventory problems as mixed-integer programming models.

Reorder policies are first determined for the 750 problems by assuming that all periodic demands are known with certainty. These policies, based upon perfect demand information, provide a means for placing an estiamted lower limit on the expected total cost over each four-period planning horizon. The policies also provide a range of values over which reorder policies may be expected to vary for any given planning horizon. Solutions to the mixed-integer programming formulations are obtained through the implementation of a readily available, easy-to-use computer algorithm.[5]

[5]The branch-and-bound integer programming algorithm employed in this research is described on page 242 of Catalog of Programs for IBM System 360 Models 25 and Above, GC20-1619-8; program numbers 360D-15.20.005, International Business Machines Corporation.

To develop an expected total cost function for a given four-period planning horizon, twelve reorder quantities are selected over the relevant range of first-period reorder policies. An additional constraint is then added to the mixed-integer programming formulation that permits restricting the first-period reorder policy to each of the twelve reorder quantities, in turn. By restricting the first-period policy to a specific quantity, X_1, and solving the mixed IP model for each of the 30 sets of periodic demands, a researcher can estimate the expected total cost over the planning horizon. This estimate is based on implementing X_1 in the first period and optimal reorder policies contingent on the first-period policy) in each of the following three periods to solve a sample of 30 randomly generated problems. When a sufficiently large number of expected total cost estimates is computed, it is possible to graph expected total cost over a four-period planning horizon as a function of the first-period reorder policy.

The decision to accept or reject Hypothesis 1, that the expected total cost function does not have a unique minimum, is based upon an examination of the plotted expected total cost curves. Acceptance or rejection of Hypothesis 2 must be based on a subjective analysis of the computer processing time required to map an expected total cost function adequately.

Adequacy of Mixed-Integer Programming Solutions When Expectations are Used as Demand Inputs

The methodology of the study requires an assessment of the adequacy of first-period reorder policies that are obtained when expected values are used as periodic demands in the mixed-integer programming formulation. It is therefore necessary to use the computer algorithm to determine a first-period reorder policy, X_1^*, for each of the 25 sample planning horizons when expected periodic demands are used as demand inputs. Expected total cost for each planning horizon can then be estimated by restricting the first-period reorder policy to X_1^*, and solving the model again for each of the thirty sets of periodic demands.

An analysis of variance (single-factor, repeated measures) is then conducted, at the .01 level of significance, to test the validity of Hypothesis 3. The contention of Hypothesis 3 is that the expected total cost resulting from the implementation of X_1^* does not differ significantly from the expected total cost incurred when the optimal first-period policy, X_1^{opt}, is implemented. Specifically, an attempt is made to reject the null hypothesis, $ETC(X_1^*) = ETC(X_1^{opt})$.

The adequacy of mixed-integer programming solutions, given X_1^* first-period policies, is further explored by developing an interval estimator for the percentage, π, of all four-period planning horizons (subject to the postulated conditions) in which $ETC(X_1^*)$ exceeds $ETC(X_1^{opt})$ by no more

than 3 percent of ETC(X_1^{opt}). This objective is accomplished by constructing an asymmetrical 5 percent confidence interval for π.[6]

Analysis of Comparative Model Performance

A major concern of the study is to evaluate the performance of the mixed-integer programming model relative to the performances of the EOQ, (t_p,S), and deterministic (s,S) models when each is used to solve a four-period, finite horizon inventory problem characterized by beta-distributed periodic demands. Accordingly, an experiment is conducted whereby each model is implemented to solve a randomly generated sample of 25 inventory problems. Each problem is designed to require first-period replenishment decisions during 20 consecutive simulated four-period planning horizons.

The experiment requires the generation of 575 generalized beta distributions. A beta-distributed periodic demand is subsequently generated from each distribution. A set of costs, consisting of a holding cost, a reorder cost, and a stockout cost, must also be randomly generated for each of the 25 problems. The beginning inventory level for

[6]The confidence interval for a percentage should not be symmetrical about the sample percentage, p, except in the special case where p = 50 percent. This is due to the fact that the standard error of a percentage σ_p varies with the population percentage. For a given sample size, the variability of the sampling distribution of sample percentages is greater when the population percentage is closer to 50 percent than when it is not.

each problem is generated as a random fraction of the first-period demand.

The mixed-integer programming model uses expectations as periodic demand inputs to obtain 20 reorder policies for each of the 25 problems. Two different approaches are used in employing the EOQ, (t_p,S), and (s,S) models to obtain solutions to the 25 inventory problems. In the first approach, each model uses estimates of the most-likely demand for each period as demand inputs. In the second approach, demand inputs correspond to averages of most-likely demand estimates over the four-period planning horizons.

The periodic reorder policies determined by employing the mixed-integer programming model are provided as inputs to a computer code that is used to compute the total costs for each 20-period problem resulting from the simulated implementation of each of the four models. These total costs are used as bases of comparison in evaluating the performance of the mixed-integer programming model relative to the performance of both versions of the EOQ, (t_p,S), and (s,S) models.

An analysis of variance (single-factor, repeated measures) is next conducted to test the contention that there is no significant difference among the expected total costs incurred by using the mixed-integer programming model and each version of the three other models. This analysis entails testing, at the .01 level of significance, the null hypothesis:

ETC(IP) = ETC(EOQ1) = ETC(EOQ2) = ETC(t_p,S1) = ETC(t_p,S2) = ETC(s,S1) = ETC(s,S2); where ETC(IP) represents the expected total cost over 20 consecutive periods resulting from implementing the mixed IP model, ETC(EOQ1) represents the expected total cost resulting from using the EOQ model when demand inputs correspond to estimates of most-likely periodic demands, etc.

In the event the null hypothesis is rejected, the research methodology calls for conducting pairwise Student's *t* tests to contrast the performance of the mixed-integer programming model with that of each version of the three other models. Specifically, the methodology requires attempting to reject, at the .05 level of significance, the following six null hypotheses:

Hypothesis 4: ETC(IP) $\geq$ ETC(EOQ1)

Hypothesis 5: ETC(IP) $\geq$ ETC(EOQ2)

Hypothesis 6: ETC(IP) $\geq$ ETC(t_p,S1)

Hypothesis 7: ETC(IP) $\geq$ ETC(t_p,S1)

Hypothesis 8: ETC(IP) $\geq$ ETC(s,S1)

Hypothesis 9: ETC(IP) $\geq$ ETC(s,S2)

The choice of the pairwise *t* test as a follow-up test to the analysis of variance is attributed to the fact that it is a test of the significance of the difference between two *dependent* sample means. Obviously, a statistical test, such as the standard two-sample *t* test, that assumes independence between samples would be inappropriate in this

case, since both models use exactly the same demand and cost data to arrive at periodic reorder policies. The pairwise *t* test offers another advantage over the standard two-sample *t* test in that it is not necessary to assume that the variances of the two samples are equal.

Rejection of any of the nine hypotheses relating to comparable model performances, given the experimental conditions, implies that the mixed-integer programming model performs at least as well as the model specified by the hypothesis.

The study is further concerned with determining whether the relative performances of the inventory models are significantly affected by the ratio of stockout cost to holding cost. Accordingly, linear regression techniques are employed to test the sensitivity of observed expected total cost differences to changes in the value of the cost ratio.

CHAPTER III

MODELS STUDIES IN THE RESEARCH

Introduction

This chapter provides a discussion of the four models evaluated in the research, as well as the specification of how each model can be implemented over a finite multiperiod planning horizon when the decision maker is able to revise inventory policies based on current realizations.

The Mixed-Integer Programming Inventory Model

A major objective of this research was to study the appropriateness of formulating the multiperiod inventory control problem as a deterministic mixed-integer programming model. Although a search of the literature revealed no instances in which a mixed IP model has actually been employed as an aid to inventory control, the model would appear to offer a number of distinct advantages to the practitioner. For example, the mixed IP model can be formulated to accommodate (a) costs that are variable over time, (b) nonconstant demand rates, (c) replenishment that occurs either instanteously or at a constant rate throughout a period, (d) limitations on storage and/or production capacity, (e) funds-flow restrictions, (f) inventory taxes, and (g) multiple vendors.

Another apparent advantage stems from the fact that the relevance of integer programming to planning and budgeting is well established [Jensen, 1968]. This fact suggests that, when an inventory problem is formulated in an integer-programming format, the inventory problem appears in a form that can be embedded within a more comprehensive corporate budgeting or planning model. The model thus affords the opportunity for integrating inventory management with corporate strategy.

Underlying assumptions of the mixed IP model. For the purposes of this study, a deterministic mixed-integer programming model is formulated to solve inventory problems characterized by the following assumptions:

1. The planning horizon is finite and can be partitioned into n (finitely many) periods.

2. Intraperiod withdrawal of items from inventory can be approximated with acceptable error by a constant intraperiod demand rate.

3. Shortages that occur during a given period will not be made up in subsequent periods.

4. In the event a replenishment occurs during a period, the replenishment occurs at the beginning of the period.

These assumptions are introduced to reduce slightly the complexity of the IP model developed in this study. The

assumptions may be relaxed in order to model other inventory problems more accurately.

<u>Mixed IP model notation.</u> The following notation is used in the development of the multiperiod mixed IP inventory model.

j = 1, 2, ..., n, denotes periods.

d_j = demand (number of units) during period j.

r_j = cost of replenishment (independent of amount) during period j.

s_j = cost per unit stockout during period j.

h_j = cost of holding one unit in inventory during period j.

b = beginning inventory level (just prior to start of period 1).

X_j = units of replenishment stock to be obtained at the start of period j.

W_j = units of stock withdrawn from inventory during period j.

Δ_j = 1 if $X_j > 0$; 0 otherwise.

The X_j are decision variables that represent inventory policy. When assigned numerical values, the X_j indicate when a replenishment is to occur and the magnitude of the replenishment. Stockouts occur whenever $W_j < d_j$. The Δ_j are zero-one, integer-valued variables that serve to introduce fixed replenishment charges whenever the corresponding X_j are strictly positive in value.

Mixed IP model formulation. To develop an appropriate mixed IP model, it is convenient to denote, in terms of the foregoing notation, the amounts of stock on hand at the start and end of each period. Table 2 shows the balances that would be of interest when a planning horizon that has been partitioned into four periods is considered.

The following general relationships can be derived from Table 2:

$$\text{Stock on hand at start of period } J = \Sigma_{i=1}^{j} X_i - \Sigma_{i=1}^{j-1} W_i + b \tag{4.1}$$

$$\text{Stock on hand at end of period } j = \Sigma_{i=1}^{j} (X_I = W_i) + b \tag{4.2}$$

In (3.1), the summation equals zero when its upper limit is zero.

The objective function of the model takes the form

$$\begin{array}{l}\text{Minimize}\\ \text{total cost}\\ \text{over the}\\ \text{planning}\\ \text{horizon}\end{array} = \Sigma_{j=1}^{n} \begin{bmatrix} \text{reorder} & & \text{stockout} & & \text{holding} \\ \text{cost for} & + & \text{cost for} & = & \text{cost for} \\ \text{period } j & & \text{period } j & & \text{period } j \end{bmatrix}. \tag{4.3}$$

The development of expressions for periodic reorder and stockout costs is straightforward.

$$\text{Reorder cost for period } j = r_j \Delta_j \tag{4.4}$$

$$\text{Stockout cost for period } j = s_j(d_j - W_j). \tag{4.5}$$

Table 2

Formulation of Expressions for Stock on Hand

Period	Stock on Hand at Start of Period	Stock Withdrawn	Stock on Hand at End of Period
1	$b + X_1$	W_1	$b + X_1 - W_1$
2	$b + X_1 + X_2 - W_1$	W_2	$b + X_1 + X_2 - W_1 - W_2$
3	$b + X_1 + X_2 + X_3 - W_1 - W_2$	W_3	$b + X_1 + X_2 + X_3 - W_1 - W_2 - W_3$
4	$b + X_1 + X_2 + X_3 - W_1 - W_2 - W_3$	W_4	$b + X_1 + X_2 + X_3 + X_4 - W_1 - W_2 - W_3 - W_4$

Negative stockout costs are prevented by a constraint (to be defined later) that restricts the values W_j may assume.

Given the assumption that demand during period j can be adequately approximated by a constant rate of withdrawal,

an expression for the holding cost during period j may be developed in the following manner:

$$\begin{aligned}\text{Holding cost for period } j &= h_j \text{ (average inventory during period } j),\\ &= .5h_j \text{ (stock start of } j + \text{stock end of } j),\\ &= .5h_j \left[\left(\Sigma_{i=1}^{j} X_i - \Sigma_{i=1}^{j-1} W_i + b\right) + \right.\\ &\qquad \left. \Sigma_{i=1}^{j} (X_I - W_i) + b\right],\\ &= h_j \left[\Sigma_{i=1}^{j} (X_i - W_i) + (W_j/2) + b\right]. \qquad (4.6)\end{aligned}$$

The general form of the objective function is obtained by substituting (4.4), (4.5), and (4.6) into (4.3):

$$\begin{aligned}\text{Minimize total cost over the planning horizon} &= \Sigma_{j=1}^{n} \left\{ r_j \Delta_j + S_j (d_j - W_j) + h_j \cdot \right.\\ &\qquad \left. \left[\Sigma_{i=1}^{j} (X_i - W_i) + (W_j/2) + b\right]\right\}. \qquad (4.7\end{aligned}$$

Three constraints are required for each period:

(1) Amount withdrawn during period j $\leq$ stock on hand at start of period j, or

$$W_j \leq \Sigma_{i=1}^{j} X_i - \Sigma_{i=1}^{j-1} W_i + b, \quad (j = 1, 2, \ldots, n). \qquad (4.8)$$

(2) Amount withdrawn during period j $\leq$ demand during period j, or

$$W_j \leq d_j, \qquad (j = 1, 2, \ldots, n). \qquad (4.9)$$

(3) $\Delta_j = 1$ if $X_j > 0$; 0 otherwise. $(j = 1, 2, \ldots, n)$. (4.10)

To achieve a form amenable to the solution algorithm used in this research, the third constraint may be expressed equivalently as:

$$\left(\Sigma_{i=1}^{n} d_i\right) \Delta_j - X_j \geq 0, \text{ where } \Delta_j = 0 \text{ or } 1. \qquad (4.11)$$

The solution algorithm implicitly imposes nonnegativity constraints on all variables. It also contains provisions for implicit bounds on variables. Consequently, the second constraint and the condition that all Δ_j be limited to values not in excess of unity can be accommodated in an especially efficient manner.

By combining (4.7), (4.8), (4.9), and (4.11), the model can be stated as

$$\begin{array}{l}\text{Minimize}\\ \text{total cost}\\ \text{over the}\\ \text{planning}\\ \text{horizon}\end{array} = \Sigma_{j=1}^{n} \left\{ r_j \Delta_j + s_j (d_j - W_j) + h_j \left[\Sigma_{i=1}^{j} (X_i - W_i) + (W_j/2) + b \right] \right\}$$

$$\begin{array}{lll}
\text{subject to} & W_j \leq \Sigma_{i=1}^{j} X_i - \Sigma_{i=1}^{j-1} W_i + b & (j = 1, 2, \ldots, n), \\
 & \left(\Sigma_{i=1}^{n} d_i\right) \Delta_j - X_j \geq 0 & (j = 1, 2, \ldots, n), \\
 & W_j \leq d_j & (j = 1, 2, \ldots, n), \\
\text{(implicitly)} & \Delta_j \leq 1 & (j = 1, 2, \ldots, n), \\
 & \text{all variables} \geq 0 & \\
\text{and} & \Delta_j \text{ integer valued} & (j = 1, 2, \ldots, n).
\end{array}$$

Implementation of the mixed IP model over a four-period finite planning horizon. The application of the mixed IP model to an inventory problem characterized by a four-period finite planning horizon and changing periodic demands and costs is fairly straightforward. The assertion is particularly true if the practitioner has been exposed to simple linear programming models and has submitted a deck of data cards for computer processing by a simplex program. After obtaining demand and cost estiamtes for the next four periods, the practitioner merely enters this information on data cards according to format specifications provided with the solution algorithm. The computer solution subsequently obtained will provide optimal reorder policies for each of the four periods. The policy of interest (and the one to be implemented), however, is the first-period policy. The process would be repeated just prior to the start of each period. The practitioner is thus permitted to take advantage of information about the realization of events during the previous period to adjust beginning stock levels and demand and cost estimates for subsequent periods.

Although the mixed IP model that is formulated in this research is appropriate to only a relatively simple inventory problem, it can be modified by introducing additional constraints and/or variables to incorporate a host of additional, pervasive, considerations. A later section of this study is concerned with exploring alternative methods

for possibly modifying the IP model solution in order to enhance solutions obtained under conditions characterized by stochastic periodic demands.

The Basic Lot-Size (EOQ) Model

The EOQ model with constant withdrawal rate is the most basic, and best known, of all inventory models. Discussion of the theory underlying the model can be found in virtually any introductory textbook concerned with optimization techniques.[7] The model is based upon the assumption that demand will continue at a constant rate of d units per period of duration T over an infinite time horizon. Since demand is known and constant, there is no need to consider stockouts. From the practitioner's point of view, demand must persist at an unchanging rate over a sufficiently long planning horizon for an infinite-horizon model to serve as a reasonable representation of reality. This condition is not likely to be satisfied in many applications of the EOQ model.

The model further assumes that holding costs and reorder costs remain constant ad infinitum. Therefore, the model is strictly appropriate only in situations in which both prices and capital costs remain constant over the long run. The model also makes the assumption of the existence of unlimited resources.

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[7] See, for example, S. B. Richmond, Operations Research for Management Decisions (Ronald, 1968).

EOQ model assumptions. Development of the classic EOQ model is based on these specific assumptions:

1. A constant number of units per period of duration T will be required over an infinite time horizon.

2. Reorder and holding costs are unchanging ad infinitum.

3. Stockouts do not occur.

4. Inventory is depleted at a constant rate over time.

5. Replenishment occurs the instant the inventory is depleted.

EOQ model notation. The following notation is commonly used in the detivation of the EOQ model.

c_h = cost of holding one unit in inventory for an interval of duration T,

c_r = cost of placing a replenishment order (independent of size of order),

d = demand per period of duration T,

Q = reorder quantity.

The purpose of the EOQ model is to determine the value of Q that minimizes total costs.

The EOQ formula. In the EOQ model, stockouts are not permitted by assumption. Thus, total cost may be expressed simply as

$$\text{Total cost over } T = \text{Reorder cost over } T = \text{Holding cost over } T$$

$$= \frac{c_r d}{Q} + \frac{c_h Q}{2} . \qquad (4.12)$$

Differentiating this expression with respect to Q, setting the result equal to zero, and solving for Q yields the well-known expression for the economic order quantity,

$$Q = \frac{2c_r d}{2c_h} . \qquad (4.13)$$

Implementation of the EOQ model. For the purpose of this research, it is envisioned that a practitioner using the model to solve a four-period, finite horizon problem, characterized by variable demands and changing costs, would reasonably choose one of two approaches. The first approach would entail obtaining cost and demand estimates for the next period and using these values in the EOQ formula to compute the desired order quantity for that period. When stock is on hand at the beginning of a period, this amount would be deducted from the computed order quantity.

A second approach, given the same specified conditions, would be to compute the averages of estimated periodic demands and costs over the entire four-year planning horizon. These values would then be used to solve the EOQ formula.

The use by a practitioner of either of these two approaches violates the basic model assumptions of constant demand rates and unchanging costs over an infinite planning horizon.

The (t_p,S) Order-Level Model

The (t_p,S) model is a periodic reorder model in which the time interval between replenishments, t_p, is a known constant. In this model, each time a replenishment occurs, the amount of stock ordered is equivalent to whatever amount is required to bring the inventory up to a level of S units. The objective of the model is to determine an optimal value for S. The model may be formulated to consider problems in which stockout costs are time-dependent. In this study, however, interest is in the form of the model in which stockout costs depend only on the size (number of units) of shortage, and not on the time duration of a stockout.

Assumptions underlying the (t_p,S) model. The formulation of the (t_p,S) model is based upon the following specific assumptions:

1. A constant number of units per period T are required ad infinitum.

2. Reorder, holding, and stockout costs remain constant over an infinite planning horizon.

3. Stock is withdrawn from inventory at a constant rate.

4. Replenishment stock arrives at the end of each time period t_p.

The (t_p, S) model is similar to the EOQ model in that it is an unconstrained optimization model. Obviously, the remarks criticizing the validity of the underlying assumptions of the EOQ model can be applied to the (t_p, S) model as well.

<u>(t_p, S) model notation.</u> The following notation is used in the development of the (t_p, S) model.

c_h = cost of holding one unit in inventory for a period of length T,

c_s = cost of a stockout of one unit (time-independent),

c_r = cost per replenishment order,

d = demand per period of duration T,

t_p = prescribed interval of time between replenishments,

S = order level (number of units in inventory after replenishment),

b = backlog (total number of units short when a stockout occurs),

t = t_p/T.

<u>Statement of the (t_p, S) model.</u> Obviously, $S > dt$ can never be optimal since this would entail paying holding costs on extra units that are carried but never used. Similarly, a

value of S < 0 cannot be optimal since it would mean always having to pay avoidable stockout costs. It follows that an optimal (t_p, S) policy can result only when $0 \leq S \leq dt$. The development of the expression for the value of S that minimizes total costs is readily available in many textbooks.[8] It basically involves expressing total cost as

$$\begin{array}{l} \text{Total cost} \\ \text{during T} \end{array} = \begin{array}{l} \text{Reorder} \\ \text{cost} \end{array} + \begin{array}{l} \text{Holding} \\ \text{cost} \end{array} + \begin{array}{l} \text{Stockout} \\ \text{cost} \end{array}$$

$$= \frac{c_r}{t} + \frac{c_h S^2}{2dt} + c_s \frac{S}{\overline{d-t}} \quad (0 \leq S \leq dt) \quad (4.14)$$

and then using simple differential calculus to obtain

$$S_{opt} = \begin{cases} c_s d/c_h & \text{when } c_s < c_h t \\ dt & \text{when } c_s \geq c_h t. \end{cases} \quad (4.15)$$

<u>Implementation of the (t_p, S) model over a four-period finite planning horizon</u>. An important premise of this study is that a practitioner employing the (t_p, S) model to solve a four-period, finite,horizon problem, characterized by changing costs and variable periodic demands, would select from one of two alternative approaches. One approach would involve using next-period cost and demand estimates as model inputs to compute the optimal value for S. A new optimal S

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[8]See, for example, Roger D. Eck, <u>Operations Research for Business</u> (Wadsworth, 1975).

value would be computed for each period of the planning horizon.

The alternative approach would entail the computation of average periodic demands and costs, based on estimated demands and costs for the next four periods, and using these values as inputs to the model. Obviously, the use by the practitioner of either of these approaches, given the specified conditions, is strictly inappropriate since it constitutes violation of the basic model assumptions of constant demand rates and unchanging costs over an indefinite horizon.

The Deterministic (s,S) Model

The (s,S) model differs from both the EOQ model and the (t_p,S) model due to the fact that its solution involves finding optimal values for two unknown (or unprescribed) decision variables. The first of these, s, represents the inventory level at which replenishment occurs. The second variable, S, represents the upper inventory level. The practitioner using the (s,S) model seeks values for s and S that will minimize total inventory costs over a period of duration T. The model may be formulated to consider stock-out costs that are either time-dependent or time-independent. This study is limited to the consideration of the form of the model in which stockout costs are time-independent.

Assumptions of the deterministic (s,S) model. Specific assumptions upon which the (s,S) model is based are:

1. Holding, reorder, and stockout costs are constant ad infinitum.

2. Demand per period of duration T is constant over an infinite planning horizon.

3. Stock is withdrawn from inventory at a constant rate.

4. Replenishment occurs when the inventory level is reduced to s.

5. Replenishments bring the inventory up to level S.

(s,S) model notation. Development of the (s,S) model commonly makes use of the following notation:

c_h = cost of holding one unit in inventory for a period of length T,

c_s = cost per unit stockout,

c_r = cost per reorder,

d = demand per period of duration T,

Q = amount ordered when replenishment occurs.

$Q = S - s.$

Statement of the (s,S) model. The development of expressions for obtaining optimal values for S and s is covered in considerable detail in numerous texts.[9] By applying

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[9]See, for example, F. S. Hillier and G. J. Lieberman, Introduction to Operations Research (Holden-Day, 1974).

differential calculus techniques to expressions representing total cost, it can be shown that

$$S_{opt} = \begin{cases} \dfrac{2c_h c_r d}{c_s(2c_h - c_s)} & \text{when } c_h/c_s > 1, \\ d & \text{when } c_h/c_s \leq 1, \end{cases} \qquad (4.16)$$

and

$$S_{opt} = \begin{cases} \dfrac{2c_h c_r d}{c_s(2c_h - c_s)} & \text{when } c_h/c_s > 1, \\ d & \text{when } c_h/c_s \leq 1. \end{cases} \qquad (4.17)$$

<u>Implementation of the (s,S) model over a four-period finite planning horizon</u>. For the purpose of this study, it is theorized that a practitioner using the (s,S) model to solve a four-period, finite horizon problem, characterized by changing costs and variable demands would behave in one of two ways. The practitioner could reasonably choose to use estimates of next-period demand and costs in applying the model to find optimal values for the decision variables. He may prefer, however, to use four-period averages as cost and demand parameters.

The practitioner using the (s,S) model is not restricted to placing replenishment orders only at the beginning or end of a period. Instead, any time the stock level falls to the level s, replenishment occurs to bring the level back up to S. This feature of the model, while probably advantageous from a holding cost point of view,

has the disadvantage of resulting in higher bookkeeping and inventory monitoring costs.

When employing the (s,S) model under the conditions specified in the research, the practitioner must recognize that the important model assumptions of constant demand and costs over an infinite horizon are being violated.

CHAPTER IV

RESULTS: HYPOTHESES 1-3

Introduction

The purpose of this chapter is to report the results of analyses conducted to test the first three hypotheses of the study. The first hypothesis is concerned with the feasibility of determining an optimal first-period reorder policy, given a four-period planning horizon characterized by changing costs and beta-distributed periodic demands. The second hypothesis is concerned with the economy with which such an optimal first-period policy can be identified. The third hypothesis is tested in order to assess the adequacy of first-period policies derived from the mixed-integer programming model when expectations are used as periodic demand inputs.

The analyses conducted in this chapter utilized 100 randomly generated generalized-beta distributions and 100 randomly generated sets of inventory costs to form 25 four-period planning horizons. Each set of inventory costs was comprised of a reorder cost, a holding cost, and a stockout cost. Each generalized-beta distribution represented the demand pattern during one period. Thirty beta-distributed demands were generated from each distribution.

The generation of demand distributions and sets of inventory costs is described in detail in Chapter III. The information specifying the first planning horizon generated is presented in Table 3.

Reorder policies based on perfect demand information. The beginning inventory level and cost information presented in Table 3, together with a set of four beta-distributed periodic demands, comprise the input parameters necessary to formulate a four-period inventory problem as a mixed-integer programming model. Thirty first-period reorder policies were determined for Planning Horizon 1 by using the mixed IP computer algorithm to solve the model for each of the 30 sets of beta-distributed periodic demands. These policies were optimal policies, since they were based on actual, or perfect, demand information.

The expected total cost over Planning Horizon 1, given perfect demand information, was estimated by computing the costs incurred as a result of implementing each of the 30 policies. The first-period reorder policies and corresponding costs over the planning horizon are presented in Table 4.

The foregoing procedure was used to obtain 30 reorder policies, based on perfect demand information, for each of the 25 four-period planning horizons. Given these policies, an estimate of expected total cost could be computed for each planning horizon. In turn, these costs

Table 3

Planning Horizon 1

Period	Greatest Possible Demand	Most Likely Demand	Least Possible Demand	Skewness	Standard Deviation	Reorder Cost	Holding Cost	Stockout Cost	Beginning Inventory Level
1	213.3	70.0	8.9	.701	34.05	20.00	3.09	5.36	32.82
2	130.9	47.1	17.4	.739	18.92	40.00	2.45	4.20	
3	34.8	33.9	30.9	.230	9.64	40.00	3.10	4.95	
4	36.6	15.2	7.4	.735	4.86	100.00	2.74	3.16	

Table 4

Planning Horizon 1, Reorder Policies and Costs
Given Perfect Demand Information

Trial	Policy	Cost (R)
1	180.0	254.39
2	20.5	390.42
3	124.0	500.98
4	17.3	347.68
5	66.0	415.57
6	58.3	402.21
7	87.4	482.53
8	49.0	443.07
9	48.1	410.11
10	97.2	437.66
11	82.2	454.73
12	20.8	340.30
13	86.2	407.28
14	0.0	353.94
15	82.2	461.98
16	65.0	419.20
17	27.2	334.22
18	44.3	382.85
19	109.0	480.87
20	29.0	349.24
21	69.7	428.38
22	0.0	292.85
23	32.9	429.62
24	117.0	472.17
25	68.3	479.20
26	112.0	524.66
27	107.0	502.23
28	104.0	495.90
29	16.7	332.23
30	0.0	317.80
Total		12,354.30

Expected total cost = $12,354.40/30 = $411.48

provided estimates of the least possible expected total costs for the 25 planning horizons.

Test of Hypothesis 1

Obviously, 30 different reorder policies based on perfect demand information are of limited value to the practitioner who is confronted with uncertainty of demand. The practitioner is interested in determining an optimal reorder policy corresponding to that single reorder quantity that results in the minimum attainable expected total cost over the planning horizon. Thus, a major concern of this research is an assessment of the feasibility of identifying such an optimal policy. This assessment was accomplished by attempting to reject Hypothesis 1: Minimum ETC $[X_1 \mid X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is not unique; where ETC $[X_1 \mid X_2^{opt}, X_3^{opt}, X_4^{opt}]$ is the expected total cost over the four-period planning horizon when X_1 is implemented in the first period and optimal policies (contingent on X_1) are implemented during each of the three remaining periods.

Hypothesis 1 was tested by investigating the functional relationship between expected total cost over a four-period planning horizon and the reorder policy, X_1, implemented during the first period. The investigation was pursued by first introducing an additional constraint to the mixed IP formulation that allowed restricting X_1 to a specific value.

A relevant range that might include an optimal X_1 was then determined for each planning horizon by observing the 30 first-period policies based on perfect demand information. A value of zero, indicating no stock replenishment, should always be included in this relevant range. In certain instances, each of the 30 reorder policies called for no replenishment during the first period. Given this situation, a relevant range for X_1 could be estimated by observing the parameters of the demand distributions.

Twelve values were next selected over the relevant range of possible first-period reorder policies for Planning Horizon 1. These values were used to construct an expected total cost function for the planning horizon. Optimal policies (contingent on X_1) were determined for the remaining three periods by restricting X_1 to the first of these 12 values, and solving the mixed IP model for each of the 30 sets of beta-distributed periodic demands. The expected total cost incurred by implementing X_1 and the three optimal policies was easily computed through a simple modification of the mixed IP computer algorithm. By restricting X_1 to each of the remaining 11 values, in turn, and repeating the solution procedure, corresponding expected total costs were obtained. Table 5 presents selected values of X and corresponding estimated expected total costs for Planning Horizon 1.

Table 5

Planning Horizon 1, Estimated Expected Total Cost, Given X_1

Policy	X_1	ETC
1	0.0	544.11
2	20.0	565.54
3	30.0	542.96
4	40.0	529.14
5	45.0	516.07
6	50.0	509.92
7	55.0	508.93
8	65.0	502.10
9	75.0	493.74
10	85.0	501.42
11	100.0	530.12
12	120.0	593.61

Plotting the values displayed in Table 5 resulted in a graphical representation of the functional relationship between the first-period reorder policy, X_1, and the expected total cost over Planning Horizon 1. Expected total cost functions for twelve representative four-period planning horizons are presented in Figures 1-12. Each expected total cost function is bounded by 95 percent confidence limits.

An examination of Figures 1-12 reveals that it is generally possible to estimate the optimal reorder policy for each planning horizon by simply noting which value of X_1 results in the lowest expected total cost. In only one case, Planning Horizon 17 (Figure 9), is the expected total cost curve sufficiently flat to cause difficulty in estimating the optimal first-period policy. Based on these results, Hypothesis 1 is rejected; and it is concluded that the

$$\underset{X_1}{\text{Minimum}}\ \text{ETC}\left[X_1 \mid X_2^{opt}, X_3^{opt}, X_4^{opt} \right]$$

is unique. Therefore, the feasibility of determining an optimal X_1 is established.

<u>Test of Hypothesis 2</u>

Rejection of Hypothesis 1 attested to the validity of using a combination of mixed-integer programming and computer simulation procedures to identify optimal first-period reorder policies. The next question to be answered was whether this procedure is an economical way to identify optimal policies. Accordingly, attention was

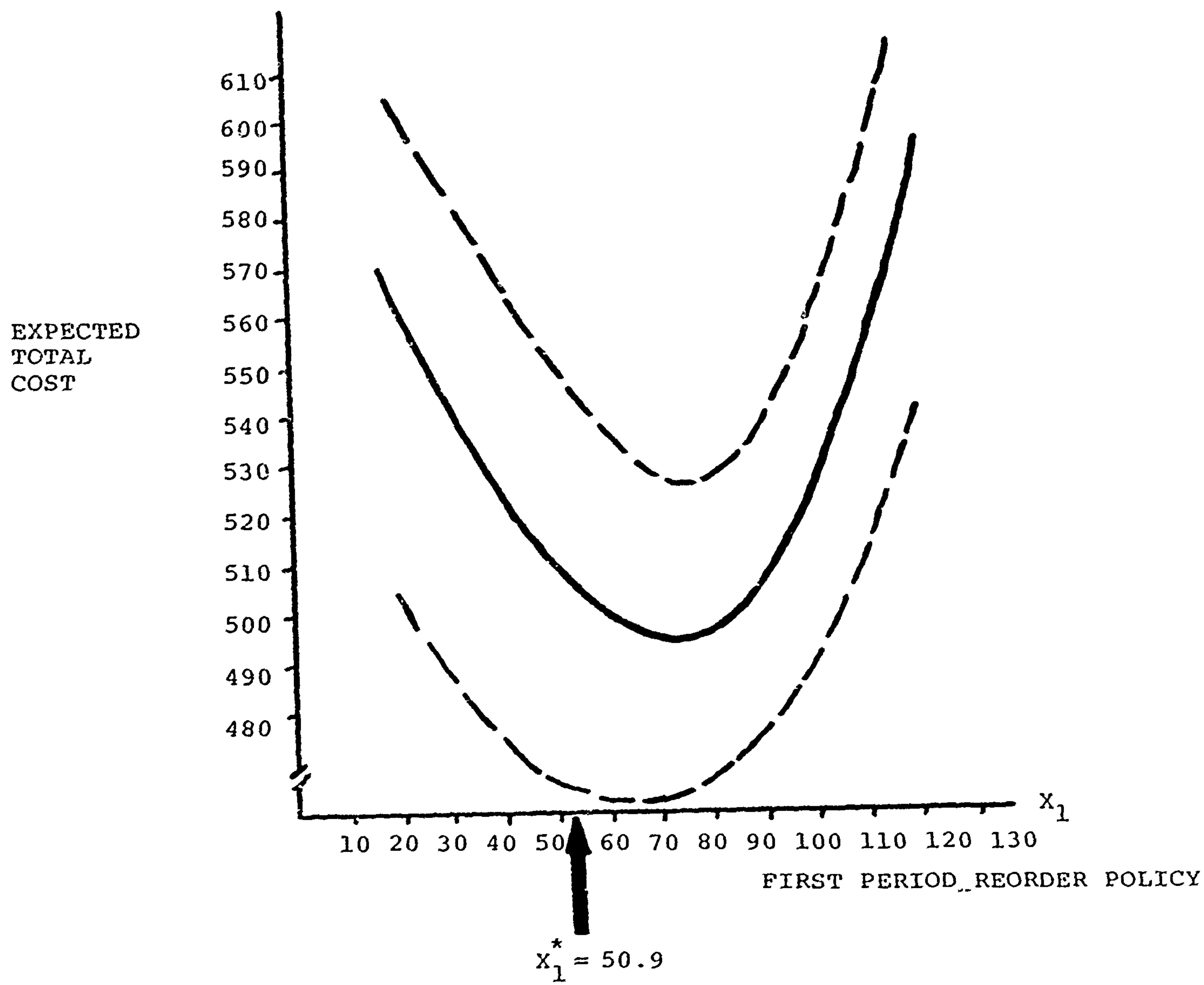

Figure 1 - Expected Total Cost Function, Planning Horizon 1

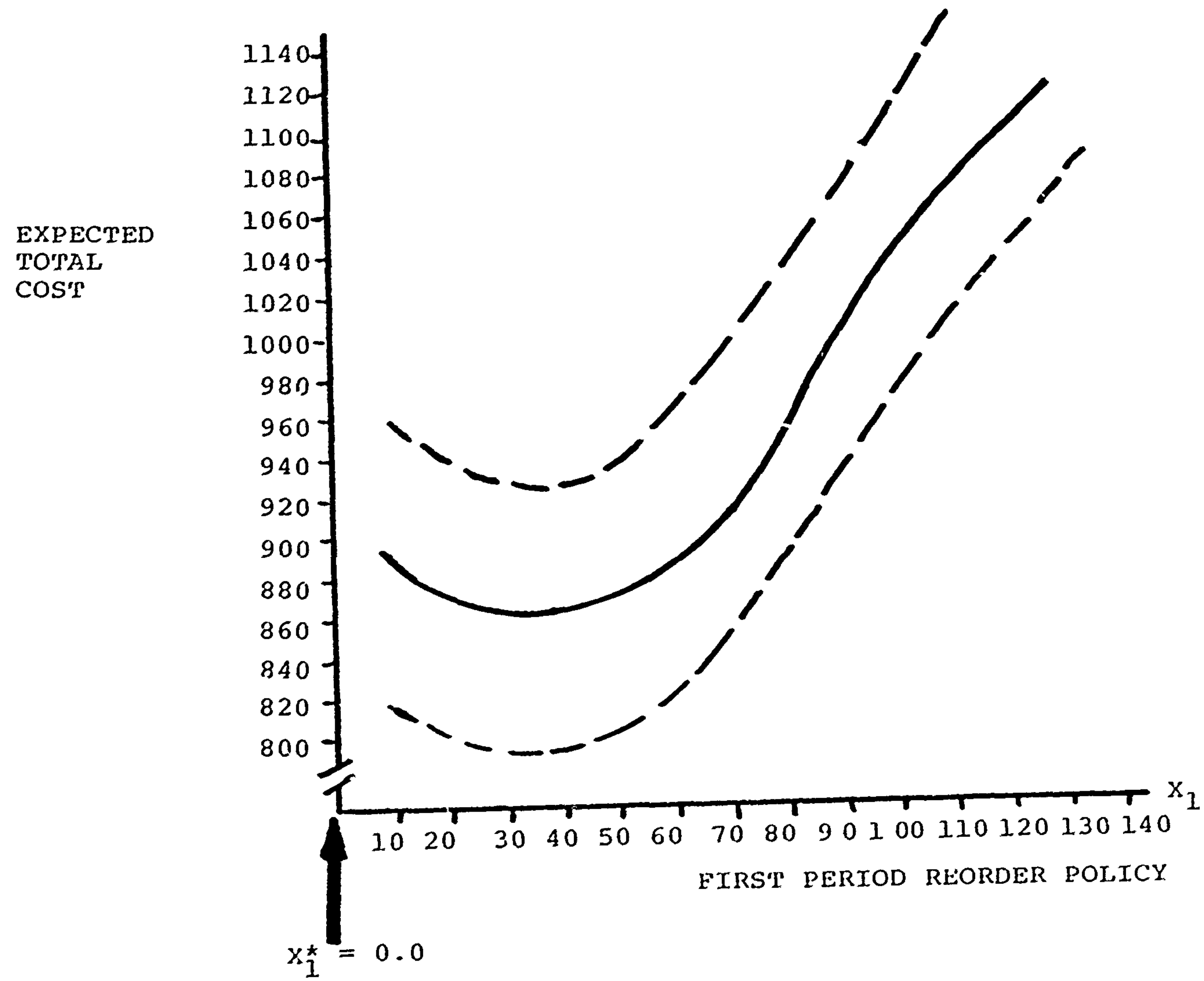

Figure 2 - Expected Total Cost Function, Planning Horizon 3

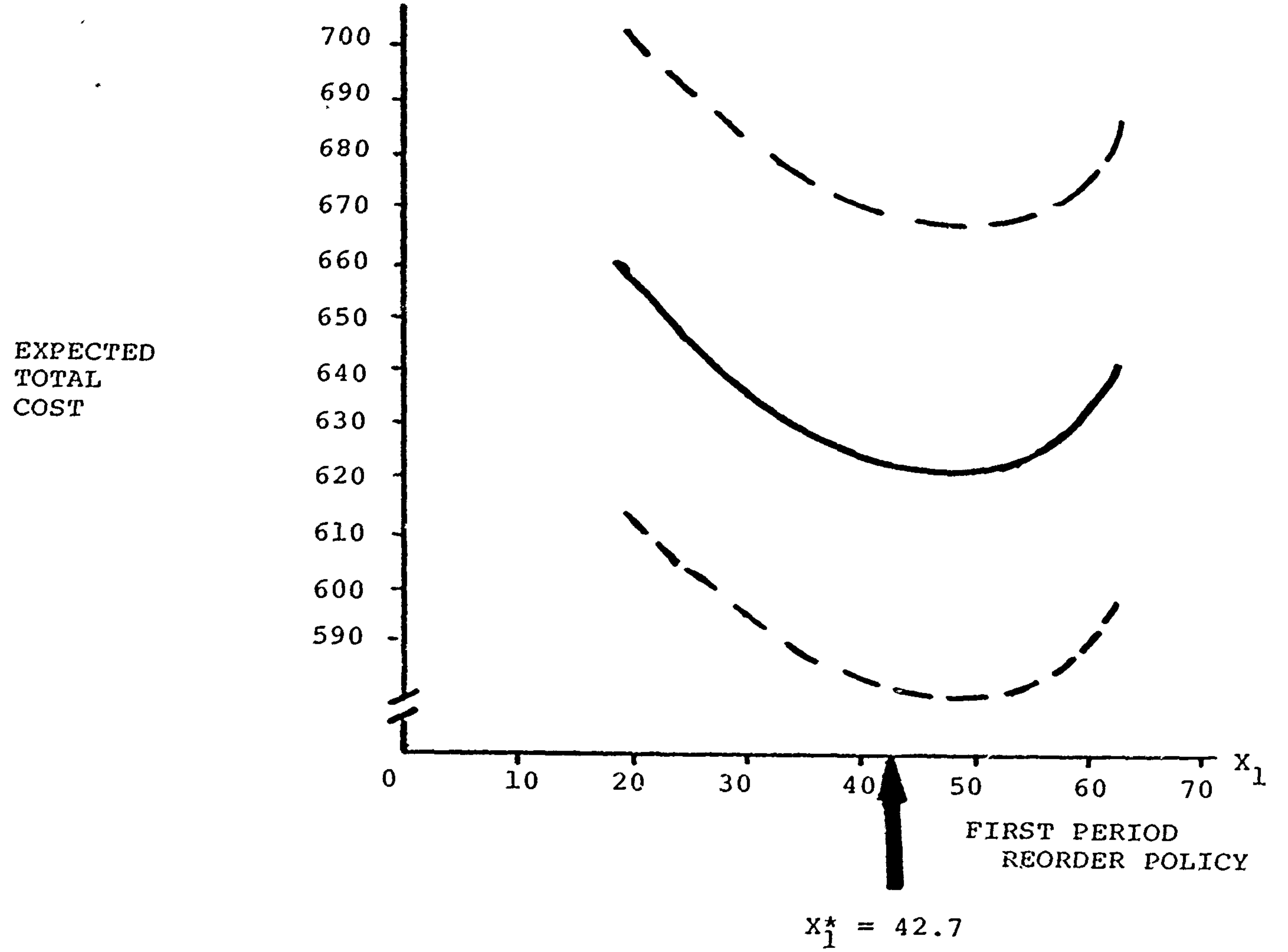

Figure 3 - Expected Total Cost Function, Planning Horizon 5

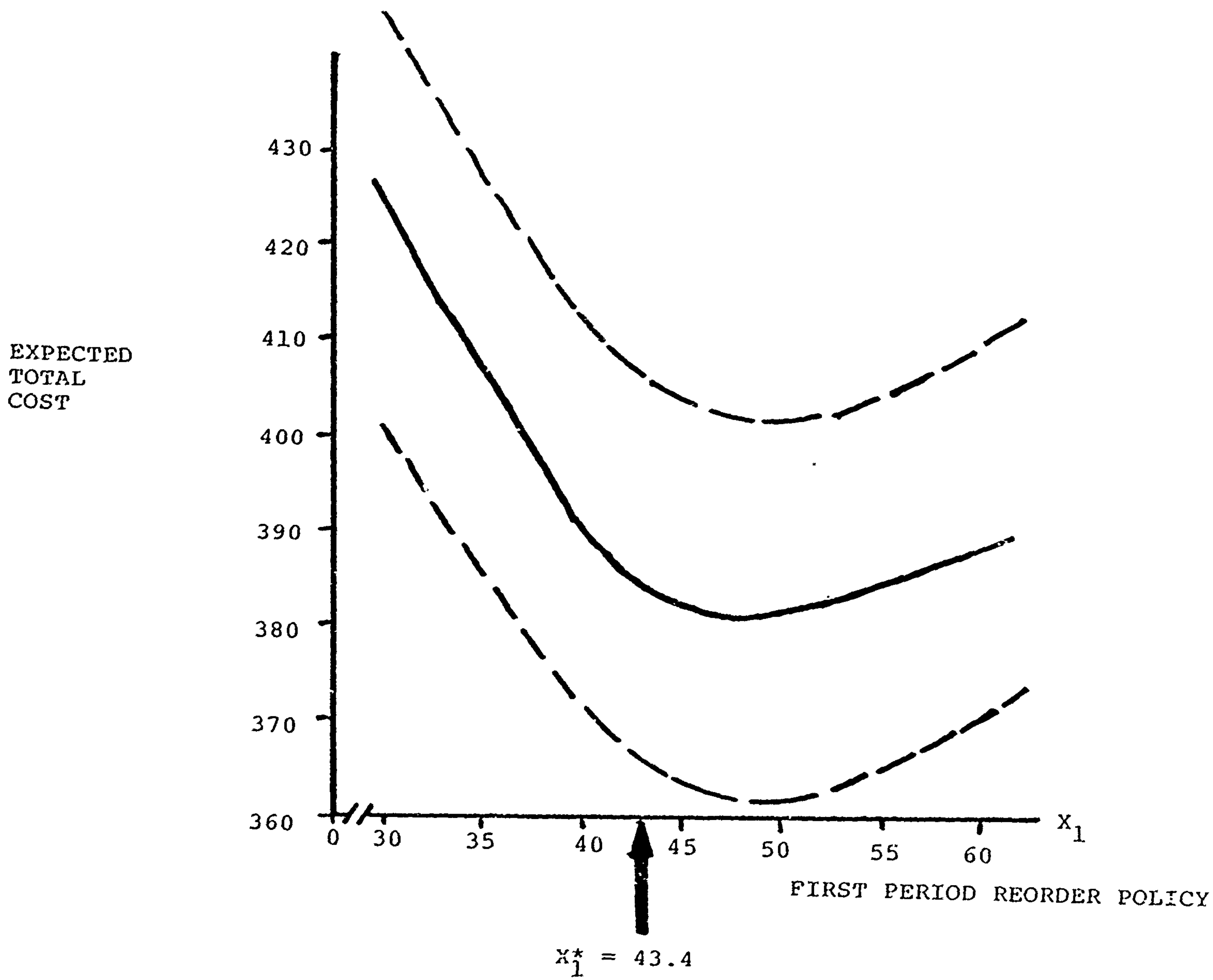

Figure 4 - Expected Total Cost Function, Planning Horizon 10

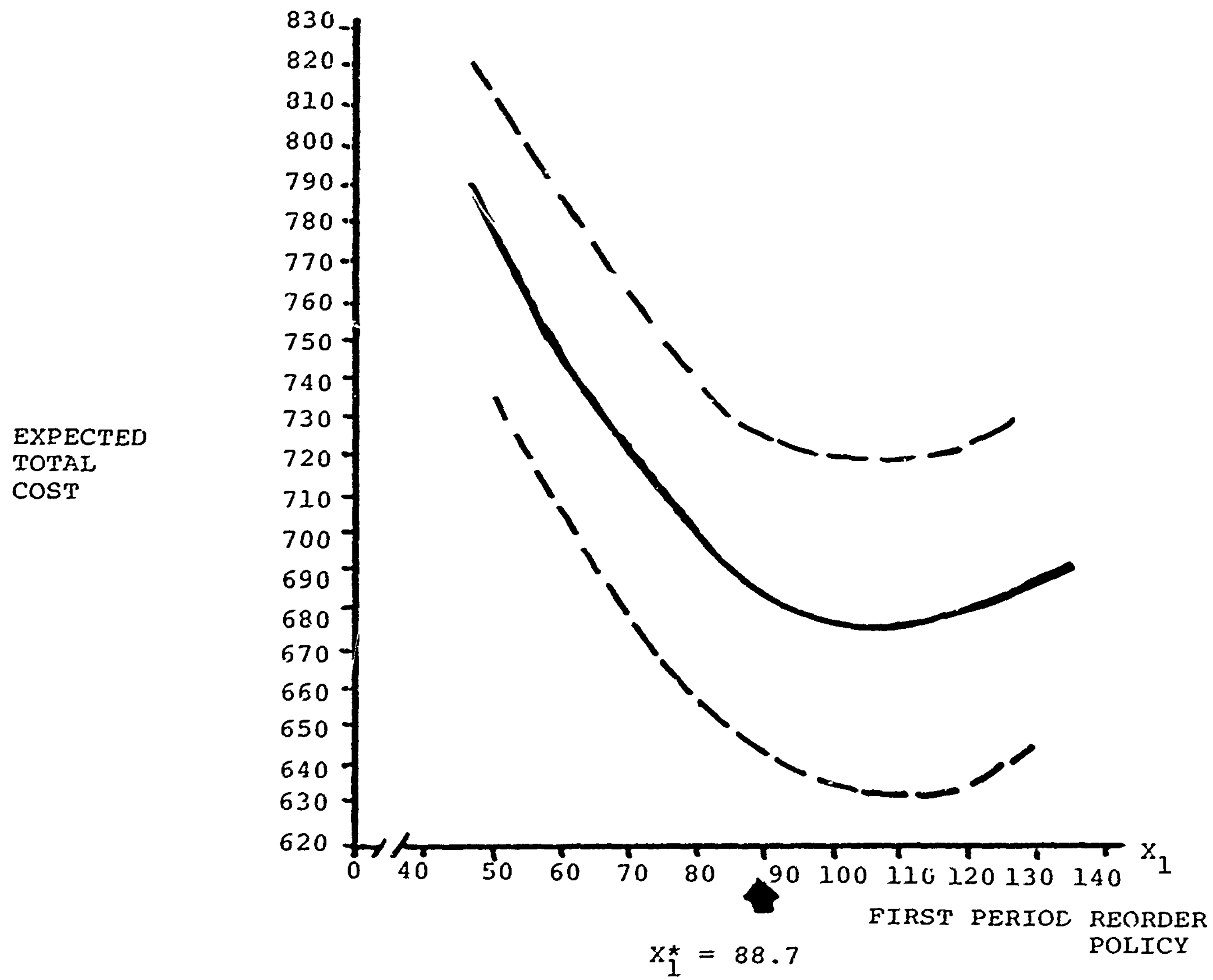

Figure 5 - Expected Total Cost Function, Planning Horizon 11

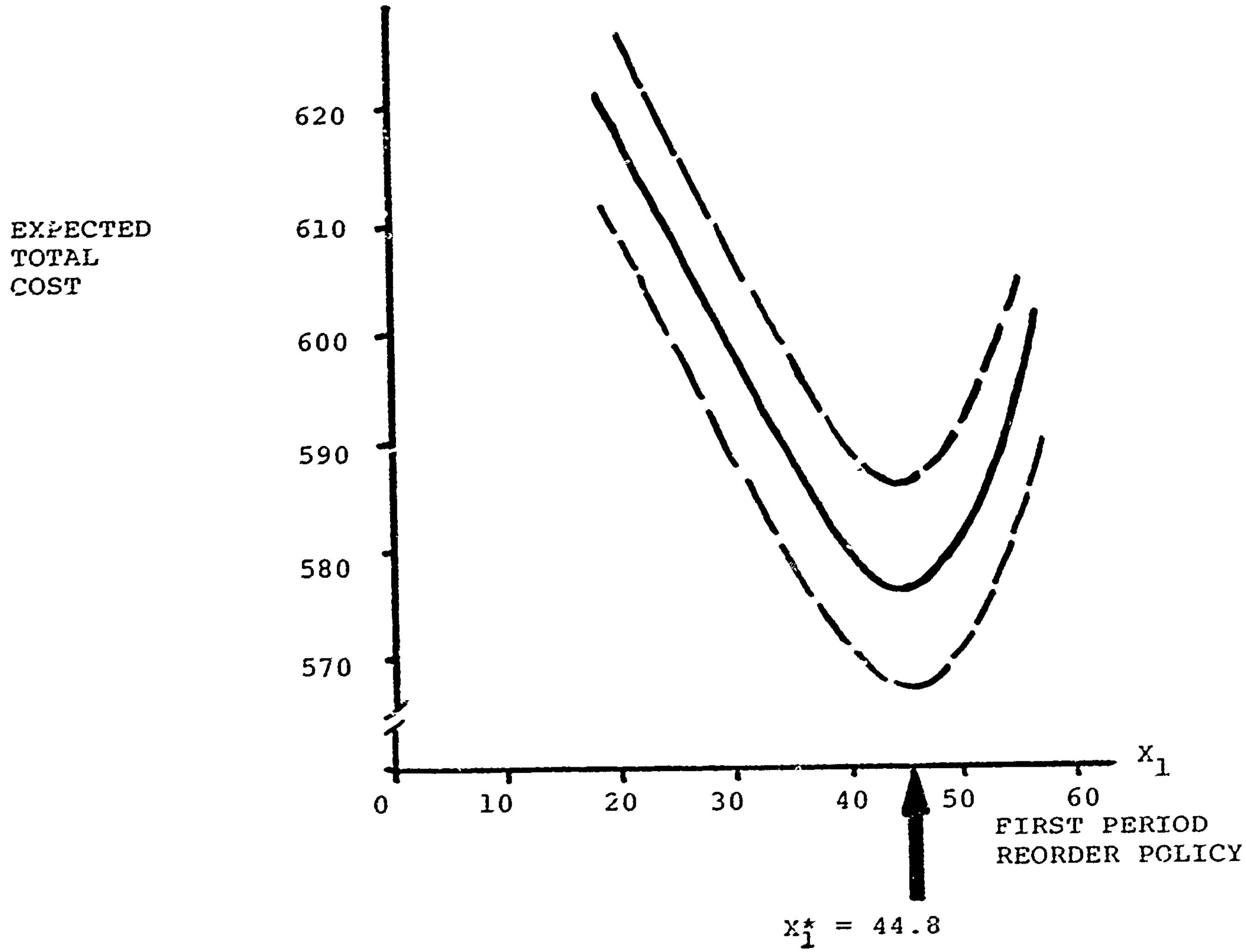

Figure 6 - Expected Total Cost Function, Planning Horizon 12

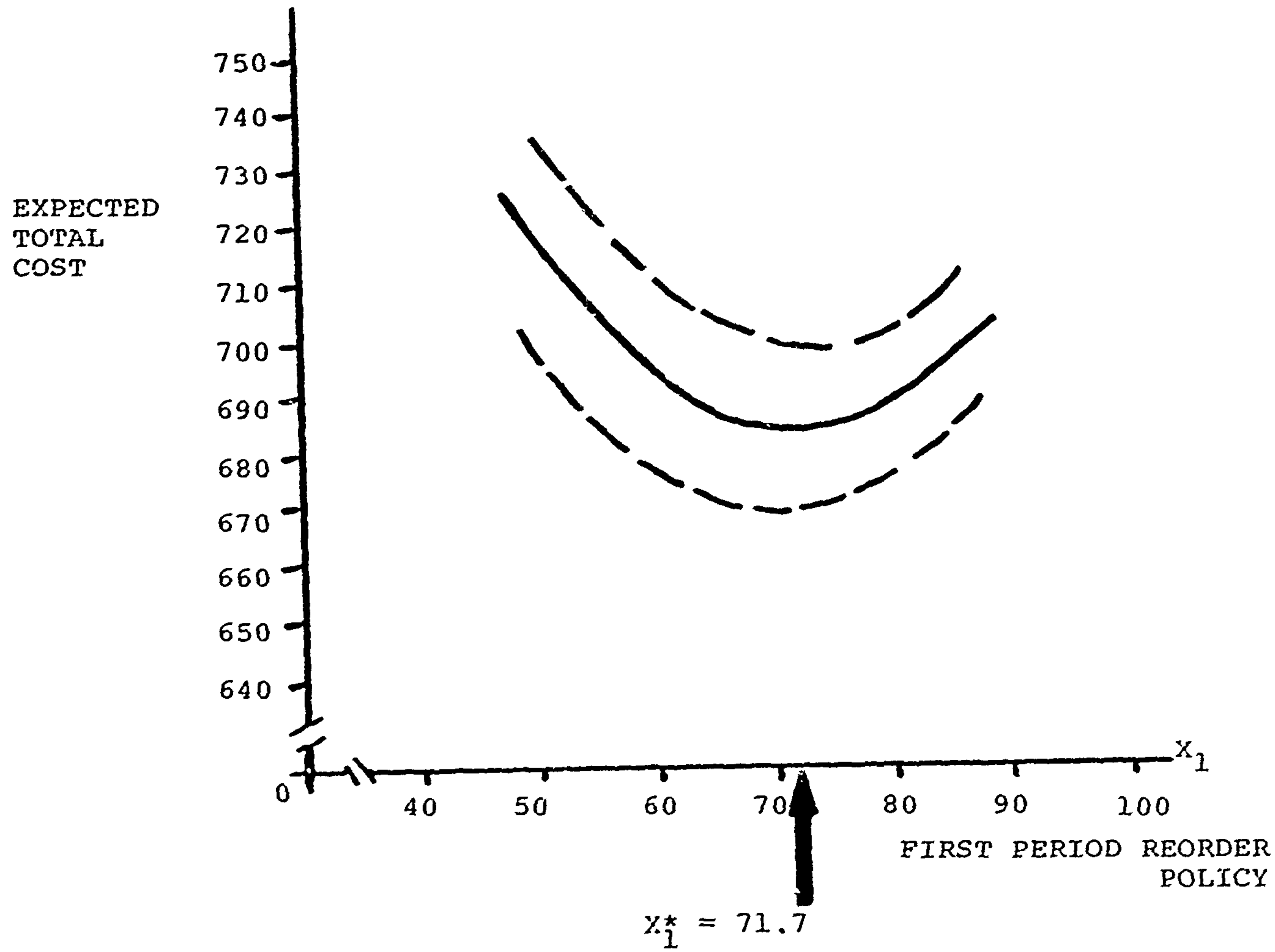

Figure 7 - Expected Total Cost Function, Planning Horizon 14

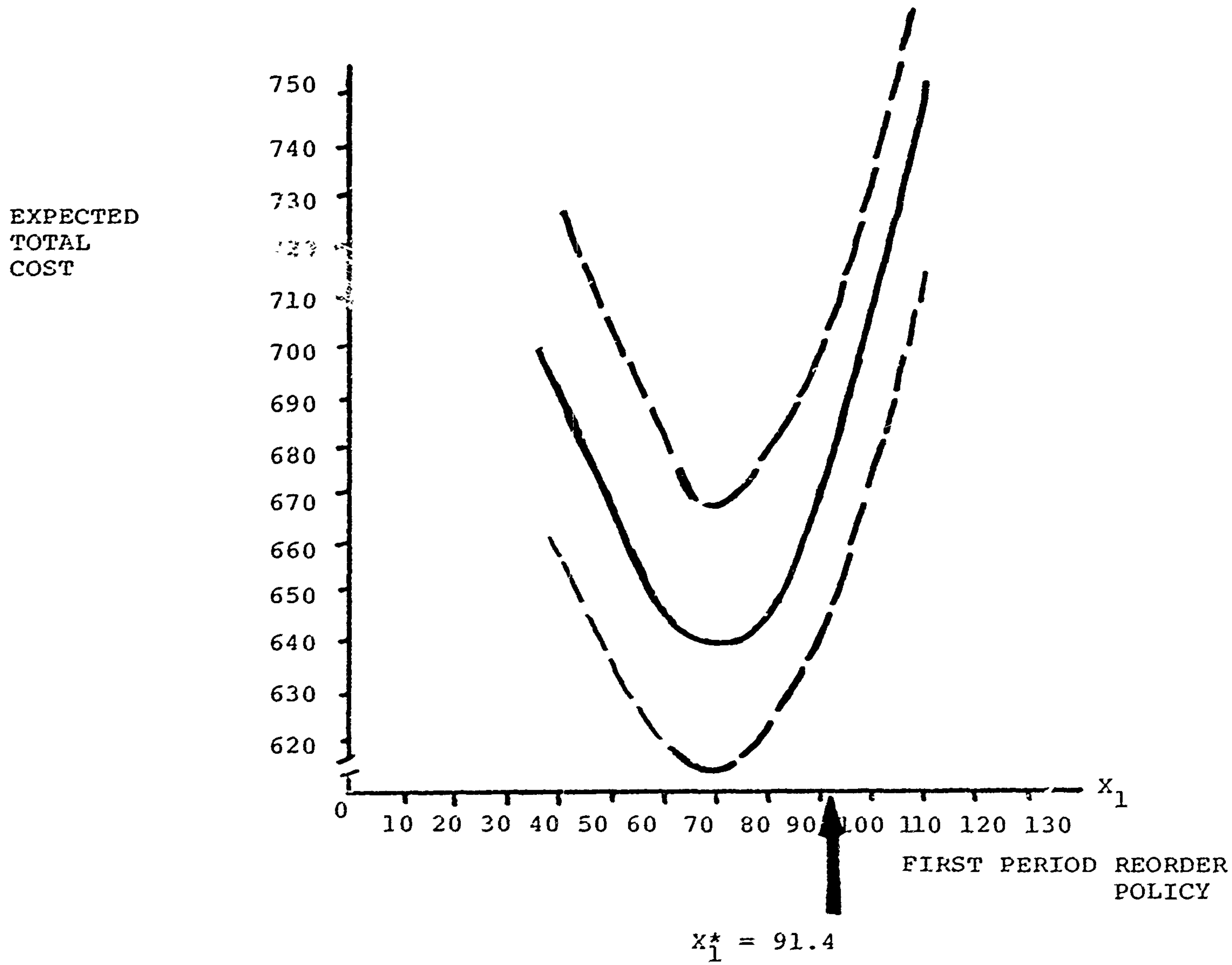

Figure 8 - Expected Total Cost Function, Planning Horizon 15

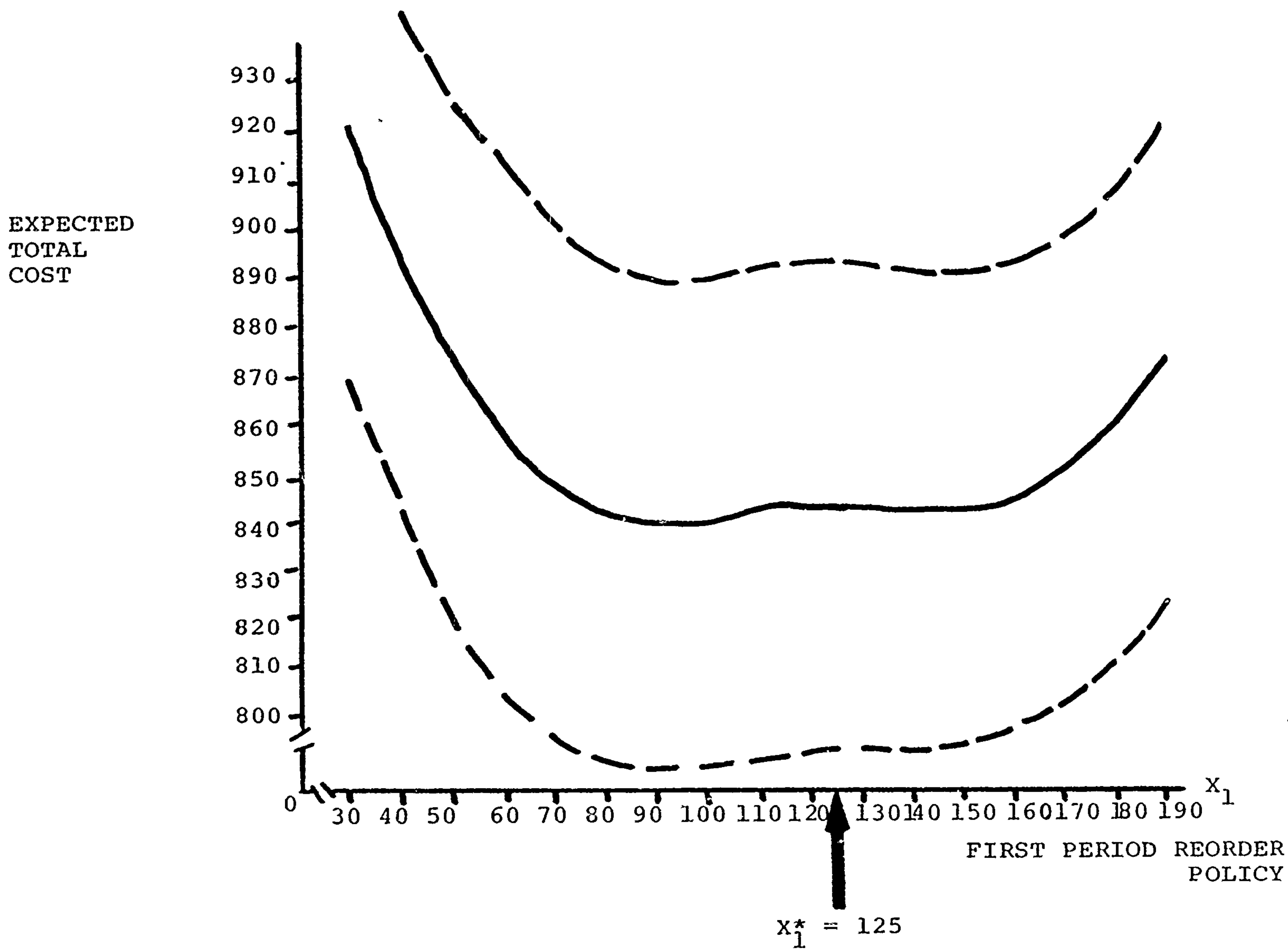

Figure 9 - Expected Total Cost Function, Planning Horizon 17

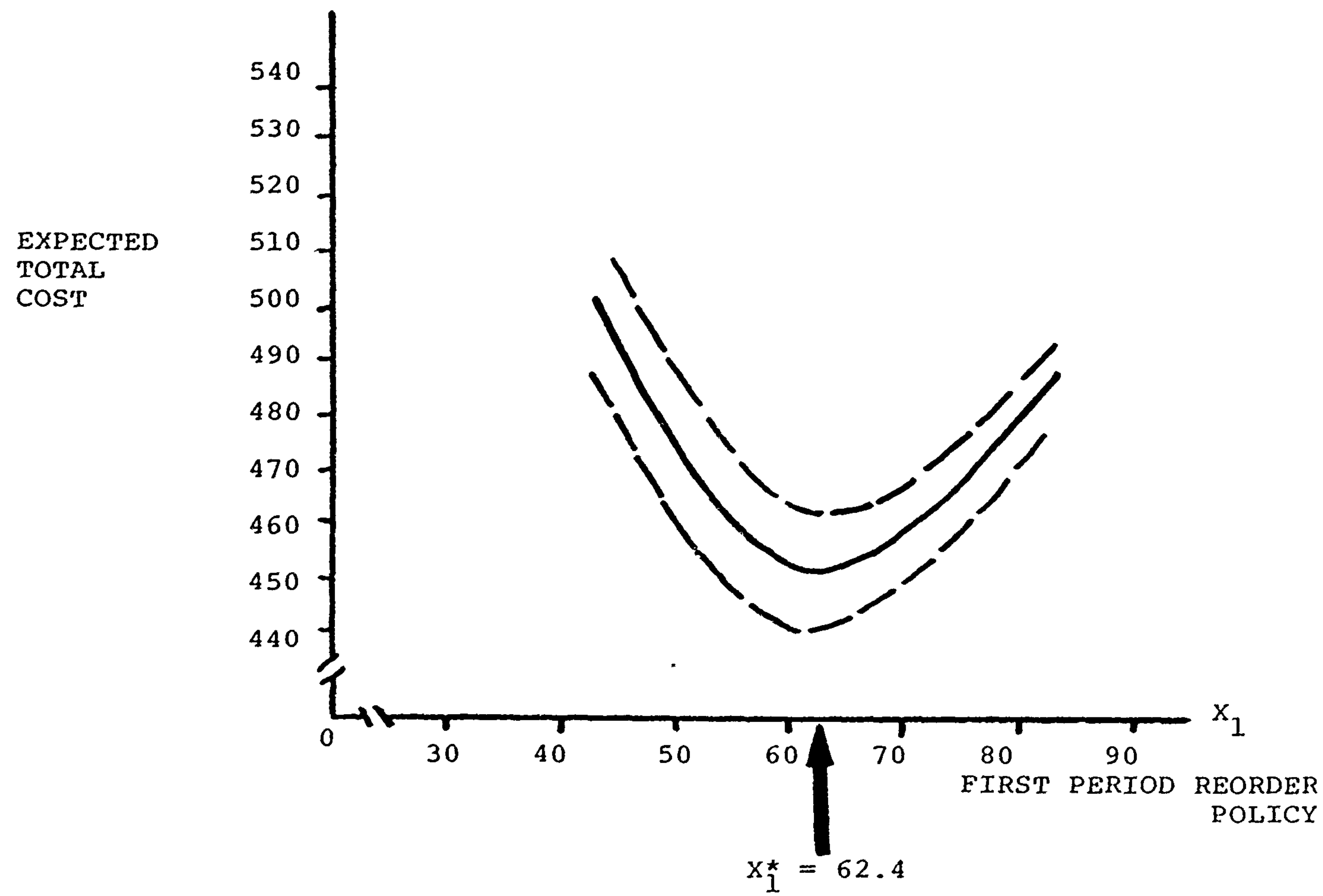

Figure 10 - Expected Total Cost Function, Planning Horizon 20

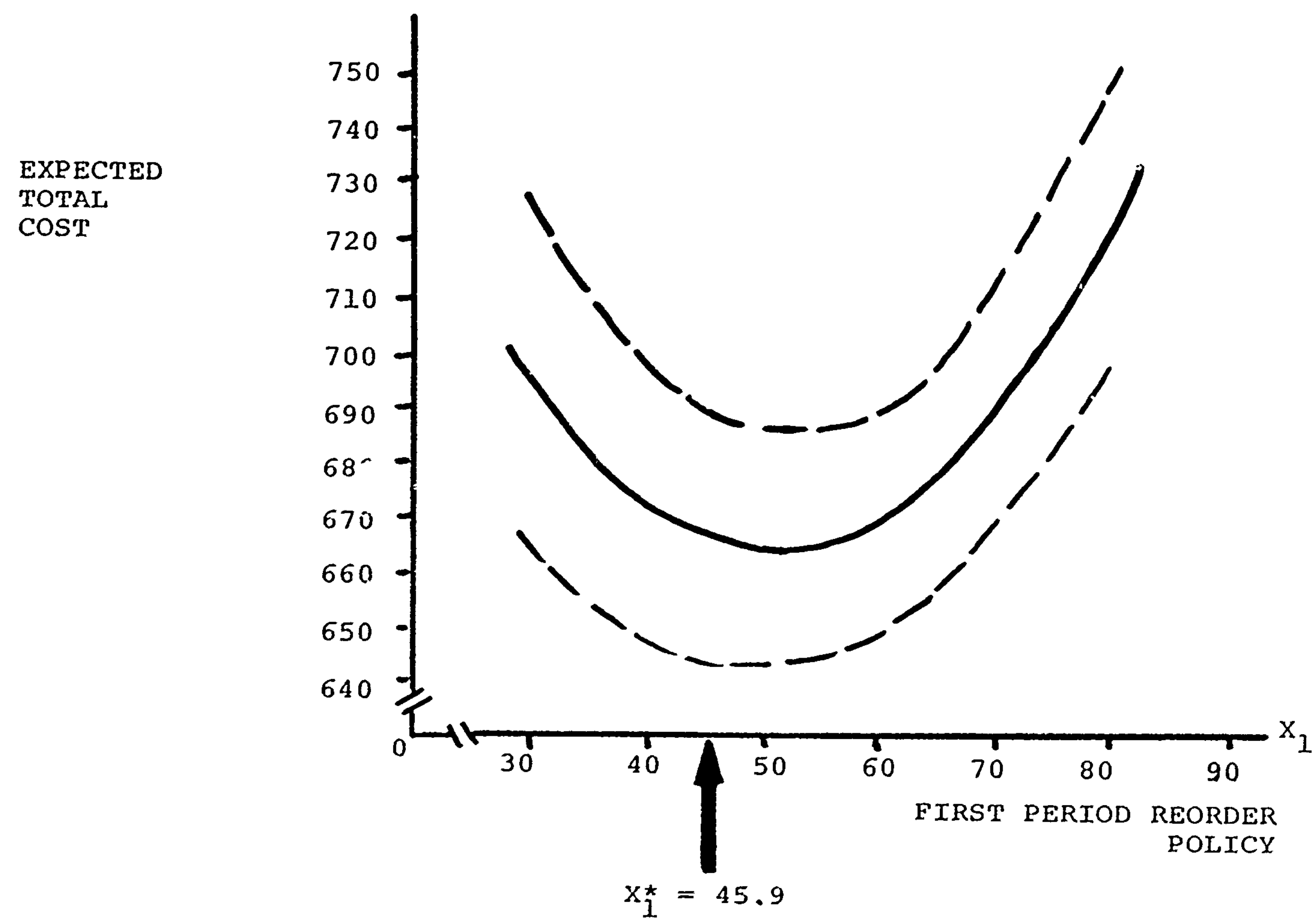

Figure 11 - Expected Total Cost Function, Planning Horizon 22

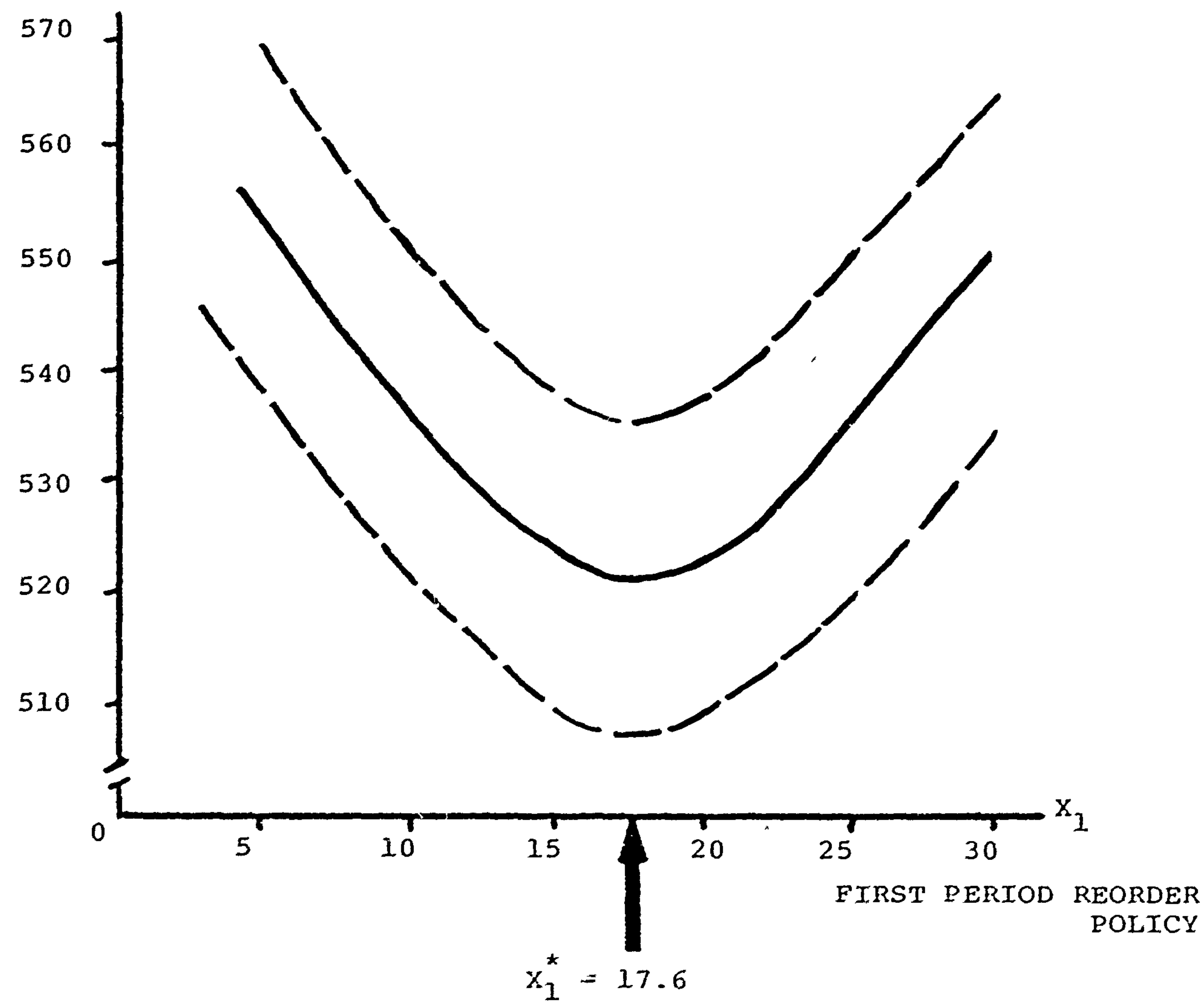

Figure 12 - Expected Total Cost Function, Planning Horizon 24

directed toward Hypothesis 2, which states: By using a combination of simulation and mixed-integer programming methods, an optimal first-period policy, X_1, can be determined over a four-period planning horizon with a nominal investment in computer processing time.

Acceptance or rejection of Hypothesis 2 was based on an analysis of the computer programming procedures and computer processing time required to map an expected total cost function adequately. Initially, these procedures require that a computer program be written that will accept the three-parameter periodic demand estimates, and use these estimates to develop the specifying parameters (α and β) required by the beta random-number generator. The computer program must then cause the generation of a sufficiently large number of beta-distributed periodic demands to permit estimates of expected total cost to be within specified limits. The development of such a computer program can be accomplished with relative ease. A similar program, developed in the course of this study to generate 40 beta-distributed demands for each of four periods, required approximately 3 seconds of computer processing time for execution by a UNIVAC 1100 series computer.

Although the basic branch-and-bound IP computer algorithm is readily available, certain modifications would be beneficial if the algorithm is to be used repeatedly. One modification entails adding statements to the basic program to provide for the direct computation of expected

total cost and the variance of the cost for each simulated first-period policy. Such a modification was included in the form of the solution algorithm used in this study and can be made very easily.

A second desirable modification would eliminate the necessity for the practitioner to transform inventory cost estimates into objective function coefficients. The relationship between the various costs and the objective function coefficients is well defined. Therefore, the task of writing a program modification that would require the practitioner to submit only the periodic cost estimates should be relatively easy. The modification would automatically convert these estimates to the forms required by the solution algorithm.

A third beneficial modification would allow sequentially restricting the first-period reorder policy to each of the trial values of X_1 prior to considering a new set of beta-distributed periodic demands. This modification simply entails establishing an additional loop within the main program and should be accomplished with relatively little difficulty.

When a UNIVAC 1100 series computer was used to execute the mixed-integer programming algorithm employed in this study, 40 four-period inventory problems (formulated according to the conditions postulated in this study) could be solved within approximately 10 seconds of computer-processing time. Thus, an analysis of 15 trial first-period

policies over 40 sets of four beta-distributed periodic demands could be accomplished in roughly 150 seconds of computer processing time. The suggested modifications to the basic mixed IP computer algorithm should result in no more than an additional 5 seconds of processing time. Therefore, given that a sample of 40 sets of random demands and 15 trial first-period policies are deemed sufficient, the identification of an optimal reorder policy should require no more than 158 seconds of computer processing time. At the current rate of $0.20 per second of computer processing time, this figure represents an investment of approximately $31.60. Given the type of company that would be interested in using such a technique as an aid to controlling its inventories, this figure appears nominal.

The relatively simple programming requirements and the nominal cost of implementation support the conclusion that an optimal first-period reorder policy can be easily and economically determined through the combined use of mixed-integer programming and computer simulation procedures. Hypothesis 2 is therefore accepted.

Test of Hypothesis 3

A primary concern of this research was an assessment of the adequacy of first-period reorder policies derived from mixed-integer programming models when expectations are used as estimates of uncertain periodic demands. This assessment was accomplished by testing, at the .01

significance level, Hypothesis 3: $ETC(X_1{*}) = ETC(X_1^{opt})$; where $ETC(X_1{*})$ is the expected total cost over a four-period planning horizon when $X_1{*}$, the first-period policy determined by the mixed IP model using expected demands, is implemented. $ETC(X_1^{opt})$ represents the expected total cost resulting from the implementation of the optimal first-period policy, x_1^{opt}. In both cases, subsequent reorder decisions over the planning horizon are assumed to be made with knowledge of the first-period policy and perfect information regarding periodic demands, and are therefore optimal.

$X_1{*}$ for each of the 25 sample planning horizons was easily determined by solving the mixed-integer programming formulation using expectations as periodic demand inputs. The introduction of an additional constraint to the basic model permitted restricting the first-period reorder quantity to $X_1{*}$ for each planning horizon. Optimal policies (contingent on $X_1{*}$) were determined for the remaining three periods of a given planning horizon by restricting the first-period policy to $X_1{*}$ and solving the mixed IP formulation for each of the 30 sets of beta-distributed periodic demands. Given this information, the task of computing the estimated expected total cost resulting from the implementation of $X_1{*}$ and X_1^{opt} were each implemented as first-period policies.

An analysis of variance was next conducted to test the contention that $ETC(X_1{*}) = ETC(X_1^{opt})$. Since both estimates of expected total cost were based upo.. identical cost

and demand information, they should be highly correlated. Therefore, the repeated measures analysis of variance was considered appropriate. Table 7 presents the results of the analysis of vairance conducted to test Hypothesis 3 at the .01 level of significance.

Since the computed F statistic, 4.877, does not exceed the critical F value, 7.82, Hypothesis 3 cannot be rejected at the .01 level of significance. In other words, it cannot be stated, with 99 percent confidence, that $ETC(X_1{}^*)$ differs significantly from $ETC(X_1^{opt})$.

Although Hypothesis 3 cannot be rejected at the .01 significance level, it should be noted that the computed F statistic of 4.877 indicates that the hypothesis would be rejected if a significance level greater than .035 is chosen. A further examination of Table 6, however, reveals that the average percentage by which the estimated $ETC(X_1{}^*)$ exceeds the estimated $ETC(X_1^{opt})$ is only 0.61. In addition, in only one instance is the percentage difference greater than 3 percent of the estimated $ETC(X_1^{opt})$.

Given the information in Table 6, a 95 percent confidence interval was constructed for the difference, D, between $ETC(X_1{}^*)$ and $ETC(X_1^{opt})$ expressed as a percentage of $ETC(X_1^{opt})$. Construction of the confidence interval for D is summarized in Table 8. Thus, the upper and lower 95 percent confidence limits for the percentage difference between $ETC(X_1{}^*)$ and $ETC(X_1^{opt})$ are .07 and 1.15, respectively.

Table 6

Estimated Expected Total Costs over Four-Period Planning Horizon, X_1* vs X_1

Planning Horizon	X_1*	X_1	Estimated ETC (X_1*) ($)	Estimated ETC (X_1) ($)	Difference ($)	Pct. Difference*
1	50.9	75.0	507.12	493.74	13.38	2.70
2	0.0	0.0	376.97	376.97	0.00	0.00
3	0.0	0.0	824.58	824.58	0.00	0.00
4	28.0	28.0	687.06	687.06	0.00	0.00
5	42.7	47.5	623.57	621.72	1.85	0.29
6	0.0	0.0	996.10	996.10	0.00	0.00
7	0.0	0.0	557.36	557.36	0.00	0.00
8	0.0	0.0	522.12	522.12	0.00	0.00
9	0.0	0.0	686.76	686.76	0.00	0.00
10	43.4	47.0	383.50	380.00	3.50	0.92
11	88.7	99.8	686.55	675.00	11.55	1.71
12	44.8	44.8	576.19	576.19	0.00	0.00
13	0.0	0.0	482.66	482.66	0.00	0.00
14	71.7	71.7	683.99	683.69	0.00	0.00
15	91.4	70.5	677.63	640.00	37.63	5.87
16	0.0	0.0	464.89	464.89	0.00	0.00
17	125.0	90.0	848.87	837.00	11.87	1.41
18	0.0	0.0	294.19	294.19	0.00	0.00
19	0.0	0.0	432.08	432.08	0.00	0.00
20	62.4	62.4	432.81	432.81	0.00	0.00
21	0.0	0.0	724.96	724.96	0.00	0.00
22	45.9	50.1	666.44	664.92	1.52	0.22
23	0.0	0.0	507.17	507.17	0.00	0.00
24	17.6	0.0	521.67	511.06	10.61	2.07
25	0.0	0.0	504.83	504.83	0.00	0.00
Avg.			586.80	583.13	3.67	0.61

Table 7

Analysis of Variance for Hypothesis 3

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F
Within horizons:				4.877**
Policies	168.75	1	168.75	
Residual	830.40	24	34.60	
Subtotal	999.15	25		
Between horizons ..	622,908.96	24		
Total	623,908.11	49		

**F(.99;1,24) = 7.82.

A second 95 percent confidence interval was constructed in a similar manner for π, where π is defined as the percentage of all four-period planning horizons (developed under the conditions postulated in this research) in which $ETC(X_1^*)$ exceeds $ETC(X_1^{opt})$ by less than 3 percent. The confidence interval was constructed by first noting the percentage, p, of sample planning horizons in which $ETC(X_1^*)$ is no more than 3 percent greater than $ETC(X_1^{opt})$. This value, p = (24/25)(100) = 96, was used as the entering argument in a prepared chart of confidence limits in order to determine asymmetrical confidence limits for π.[10] The lower and upper confidence limits obtained in this manner are 0.78 and 0.999, respectively.

The inability to reject Hypothesis 3 at the .01 confidence level, and the construction of the confidence intervals for D and π, attest to the adequacy of first-period reorder policies derived from the mixed IP formulation when expectations are used as demand inputs.

Summary

This chapter has presented the results of analyses conducted to test the first three hypotheses of the study. Testing Hypothesis 1 involved using a combination of mixed-integer programming and computer simulation procedures to

[10] From C. J. Clopper and E. S. Pearson, "The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial," Biometrika, 26 (1934).

Table 8

Confidence Interval for D

Planning Horizon	d^a	$d-\bar{d}$	$(d-\bar{d})^2$
1	2.70	2.09	4.37
2	.00	-.61	.37
3	.00	-.61	.37
4	.00	-.61	.37
5	.29	-.32	.10
6	.00	-.61	.37
7	.00	-.61	.37
8	.00	-.61	.37
9	.00	-.61	.37
10	.92	.31	.10
11	1.71	1.10	1.21
12	.00	-.61	.37
13	.00	-.61	.37
14	.00	-.61	.37
15	5.87	5.26	27.67
16	.00	-.61	.37
17	1.41	.80	.64
18	.00	-.61	.37
19	.00	-.61	.37
20	.00	-.61	.37
21	.00	-.61	.37
22	.22	-.39	.15
23	.00	-.61	.37
24	2.07	1.36	1.85
25	.00	-.61	.37
Total			42.38

[a] d = estimated percent difference between $ETC(X_1^*)$ and $ETC(X_1^{opt})$.

$\bar{d}$ = 0.61; S_d = 42.37/24 = .133; $s_{\bar{d}}$ = 1.33/25 = 0.26;

95% for D = 0.61 ± 2.064(.26) = 0.61 ± .54.

analyze a four-period, finite horizon inventory problem characterized by changing costs and beta-distributed periodic demands. It was concluded that the use of these procedures permits the identification of a first-period reorder policy, X_1, the implementation of which results in the minimum expected total cost over the four-period planning horizon. It was also concluded that there exists a unique optimal first-period reorder policy for any given four-period planning horizon that is subject to the conditions postulated in the research.

The test of Hypothesis 2 was accomplished through a subjective analysis of the programming and computer processing time requirements attendant to the identification of an optimal first-period reorder policy. The computer programming required to modify the basic mixed IP computer algorithm to make it more amenable to the multiperiod inventory problem was not considered to be excessive. As a result of an analysis of the computer processing time required to solve a four-period inventory problem, it was concluded that an optimal first-period policy can be determined over a four-period planning horizon with a nominal investment in computer processing time.

The adequacy of first-period policies derived from the mixed IP model when expectations are used as periodic demands over a four-period planning horizon was assessed by testing Hypothesis 3. An analysis of variance, conducted

at the .01 level of significance, disclosed that the expected total cost resulting from implementing mixed IP first-period policies could not be shown to differ significantly from the expected total cost incurred by implementing optimal first-period policies. It was concluded, therefore, that first-period policies obtained by using the mixed IP model with expectations as periodic demand inputs are generally adequate under the conditions postulated in the study.

CHAPTER V

RESULTS: AN EVALUATION OF COMPARATIVE MODEL PERFORMANCE

Introduction

A major intent of this study was to evaluate the performance of the mixed-integer programming inventory model compared to the performances of the EOQ, (t_p,S), and deterministic (s,S) models when each is used to solve a four-period, finite horizon inventory problem characterized by beta-distributed periodic demands. The much greater versatility and potential of the mixed-integer programming model would appear to justify its selection over any one of the three better-known models if it can be demonstrated that the mixed IP model performs at least as well as the other models.

One commonality of the four models under consideration is the assumption that future periodic demands are known with certainty. Of particular interest is the behavior of expected total cost when each of these deterministic models is implemented to solve multiperiod inventory problems in which periodic demands are stochastic. An experiment was therefore conducted to assess the significance of the difference between expected total cost incurred as a result of using the mixed IP model and the expected total costs resulting from the implementation of each of the three other

models when demands are beta-distributed random variables.

Description of experiment. The experiment consisted of using each of the four deterministic models to solve a sample of 25 randomly generated inventory problems. Each problem required first-period replenishment decisions during 20 consecutive simulated four-period planning horizons. Three-parameter estimates of demand were assumed to be available for the four periods of each planning horizon.

Although the mixed-integer programming model can easily accommodate inventory costs that change from period to period, the EOQ, (t_p,S), and (s,S) models all assume these costs to be constant ad infinitum. To avoid violation of this assumption, the experiment called for randomly generating a single set of costs, consisting of a holding cost, a reorder cost, and a stockout cost, for each of the 25 problems. The experiment also required the generation of 23 generalized beta distributions of demand for each problem. A beta-distributed periodic demand was then generated from each distribution. The beginning inventory level for each problem was generated as a random fraction of the first-period demand.

The mixed IP model used expected values as periodic demand inputs to obtain 20 reorder policies for each of the 25 problems. Two different approaches were used in implementing the EOQ, (t_p,S) and (s,S) models to obtain solutions to the 25 inventory problems. In the first approach,

estimates of the most-likely demand for each period were used by each model as demand inputs. In the second approach, averages of most-likely demands over the four-period planning horizons were used as estimates of periodic demands.

Given the beta-distributed periodic demands and the reorder policies obtained by using the mixed-integer programming model, a computer code was used to compute the total costs for each 20-period problem resulting from the simulated implementation of each of the four models. These total costs served as bases of comparison in evaluating the performance of the mixed-integer programming model relative to the performance of each version of the EOQ, (t_p,S), and (s,S) models.

Overall test of comparative model performance. The total costs resulting from the implementation of the mixed IP model and each version of the other three models are displayed in Table 9. These total costs were used to conduct an analysis of variance (single factor, repeated measures) to test the contention that the total costs incurred were independent of the inventory model employed. Specifically, an attempt was made to reject, at the .01 level of significance, the null hypothesis: $ETC(IP) = ETC(EOQ1) = ETC(EOQ2) = ETC(t_p,S1) = ETC(t_p,S2) = ETC(s,S1) = ETC(s,S2)$. ETC(IP) represents the expected total cost over 20 consecutive periods resulting from implementing the mixed IP model. ETC(EOQ1) represents the expected total cost

Table 9

Total Cost by Inventory Model

Prob-lem	Mixed IP	EOQ1	EOQ2	t_p,S1	t_p,S2	s,S1	s,S2
1	3569.85	4983.80	4900.60	3864.82	5044.14	4090.97	3744.69
2	2925.44	5753.80	6513.80	3116.96	4250.79	3065.23	2579.68
3	3243.62	5426.00	5459.60	3599.11	4842.69	2947.88	2924.02
4	1971.55	3774.80	4400.60	2195.62	3066.21	2048.58	1858.10
5	2737.59	4121.80	4344.80	2854.18	3895.14	2736.47	2787.28
6	3154.33	4854.00	5253.20	3425.80	3946.48	2409.00	2352.79
7	3314.70	5298.00	6092.40	3439.00	4463.80	3507.14	2709.44
8	2749.33	3497.60	3655.20	2737.31	3221.05	2339.53	2215.29
9	3693.24	5330.80	6514.00	3759.38	5045.48	3572.39	3561.65
10	2501.44	3424.20	3636.00	2340.82	2766.48	1932.89	2013.06
11	3436.72	5247.80	5287.40	3717.48	4773.56	3108.79	3255.48
12	2074.60	4714.00	4801.40	2405.99	3308.89	1802.31	1708.98
13	3211.42	4856.40	4621.00	3059.42	4217.74	3048.92	2816.66
14	3132.12	5717.00	6576.20	3429.59	4123.05	2297.89	2214.10
15	3589.86	5765.00	6181.80	4114.66	4886.17	3235.39	3224.30
16	2595.84	3373.80	3632.00	2511.85	3320.48	2062.67	1895.78
17	3869.53	4140.00	4061.80	3836.21	4314.04	3476.30	3570.00
18	3476.84	4121.40	4214.20	2896.68	3646.32	2754.04	2642.40
19	3009.72	3929.80	3820.80	3173.66	3792.63	2984.99	2909.74
20	4015.92	5246.40	5556.80	4314.70	5109.76	4050.73	3992.38
21	3743.35	4094.00	4284.40	3976.48	4297.29	3907.96	3779.55
22	3436.40	3936.00	4062.60	3452.39	4067.43	2966.84	2946.39
23	2899.19	4163.20	4131.40	2974.08	3516.56	2483.91	2564.30
24	3725.41	4019.00	4054.80	3814.58	4247.53	3647.96	3760.10
25	2528.89	2557.60	2676.00	2359.39	2811.48	1740.50	1755.03

incurred as a result of using the EOQ model when demand inputs correspond to estimates of most-likely periodic demands. ETC(EOQ2) is the expected total cost resulting from use of the EOQ model when demand inputs are based on averages of demand estimates over four periods, etc. The results of the analysis of variance are displayed in Table 10. Since the computed *F* statistic greatly exceeds the critical *F* value, the hypothesis was rejected. It was thus concluded that, given the conditions postulated in the experiment, expected total cost is *not* independent of the inventory model employed.

Pairwise tests of model performance. After the contention was rejected that expected total cost is independent of the inventory model that is employed, follow-up tests were conducted to determine those models between which there were significant cost differences. The primary purpose for conducting the experiment was to contrast the performance of the mixed-integer programming model with that of each version of the three other models. Therefore, a test was desired that would permit pairwise cost comparisons. Since the paired costs during each comparison would not be mutually independent, the pairwise Student's *t* test was chosen as an appropriate follow-up test.

Specifically, six pairwise Student's *t* tests were conducted in an attempt to reject, at the .05 level of significance, each of the following six null hypotheses:

Table 10

Analysis of Vairance: Overall Test of Difference Between Expected Total Costs

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Within problems:				77.15**
Models	94,071,070	6	15,671,845	
Residual	29,350,432	144	203,128	
Subtotal	123,071,070	150		
Between problems ...	25,971,960	24		
Total	149,394,462			

**F(.95;6,144) = 2.16.

Hypothesis 4: ETC(IP) $\geq$ ETC(EOQ1)

Hypothesis 5: ETC(IP) $\geq$ ETC(EOQ2)

Hypothesis 6: ETC(IP) $\geq$ ETC(t_p,S1)

Hypothesis 7: ETC(IP) $\geq$ ETC(t_p,S2)

Hypothesis 8: ETC(IP) $\geq$ ETC(s,S1)

Hypothesis 9: ETC(IP) $\geq$ ETC(s,S2)

The rejection of any of these six null hypotheses would imply that, given the postulated conditions, the mixed-integer programming model performs better than the contrasted model specified by the hypothesis. Results of the pairwise Student's *t* tests are summarized in Tables 11-16.

The rejection of Hypotheses 4, 5, 6, and 7 led to the conclusion that the use of the mixed-integer programming model, under the conditions specified in the experiment, results in an expected total cost that is lower than the cost incurred by using either version of both the EOQ and the (t_p,S) models. This lower expected total cost, coupled with the previously expounded potential advantages offered by the mixed-integer programming model, supports the use of the mixed IP model rather than either the EOQ or (t_p,S) models when confronted with an inventory problem subject to the conditions specified in this study.

Further comparison of the mixed IP and (s,S) models. The inability to reject Hypotheses 8 and 9 disallowed the conclusion that the mixed-integer programming model performs at least as well as the (s,S) model, given the conditions

Table 11

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(EOQ1)

Problem	D	$D-\overline{D}$	$(D-\overline{D})^2$
1	-1,413.95	- 84.35	7,115
2	-2,828.36	-1,498.78	2,246,341
3	-2,182.38	- 852.78	727,234
4	-1,803.25	- 473.65	224,344
5	-1,384.21	- 54.61	2,982
6	-1,699.67	- 370.07	136,952
7	-1,983.30	- 653.70	427,324
8	- 748.27	581.33	337,945
9	-1,637.56	- 307.96	94,839
10	- 922.76	406.84	165,519
11	-1,811.08	481.48	231,823
12	-2,639.40	-1,309.80	1,715,576
13	-1,644.98	- 315.38	99,465
14	-2,584.88	-1,255.28	1,575,728
15	-1,675.94	- 346.34	119,951
16	- 777.96	551.64	304,307
17	- 270.47	1,059.13	1,121,756
18	- 644.56	685.04	469,280
19	- 920.08	409.52	167,707
20	-1,230.48	99.12	9,825
21	- 350.65	978.95	958,343
22	- 449.60	830.00	688,900
23	-1,264.01	65.59	4,302
24	- 293.59	1,036.01	1,073,317
25	- 28.71	1,300.89	1,692,315
Total	.33,240.10		14,603,184

$\overline{D} = 33,249.10/25 = 1,329.60$

$S_D = \sqrt{14,603,184/24} = 780.04$

$S_{\overline{D}} = 780.04/\sqrt{25} = 156.01$

$$t = \frac{-1,329.60 - 0}{156.01} = -8.52$$

$t\ (.95;24) = -1.711$

Reject Hypothesis 4

Table 12

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(EOQ2)

Problem	D	$D-\bar{D}$	$(D-\bar{D})^2$
1	-1,330.75	274.19	75,180
2	-3,588.36	-1,983.42	3,933,955
3	-2,216.98	- 612.04	374,593
4	-2,429.05	- 824.11	679,157
5	-1,607.21	- 2.27	5
6	-2,098.87	- 493.93	243,967
7	-2,777.70	-1,172.76	1,375,366
8	- 905.87	699.07	488,699
9	-2,820.76	-1,215.82	1,478,218
10	-1,134.56	470.38	221,257
11	-1,850.68	- 245.74	60,388
12	-2,726.80	-1,121.86	1,258,570
13	-1,409.58	195.36	38,166
14	-3,440.08	-1,835.14	3,367,739
15	-2,591.94	- 987.00	974,169
16	-1,036.16	568.78	323,511
17	- 192.27	1,412.67	1,995,637
18	- 737.36	867.58	752,695
19	- 811.08	793.86	630,214
20	-1,540.88	64.06	4,104
21	- 541.05	1,063.89	1,131,862
22	- 626.20	978.54	957,541
23	-1,232.21	372.73	138,928
24	- 329.39	1,275.55	1,627,028
25	- 147.71	1,457.23	2,123,519
Total	-40,123.50		24,254,458

$\bar{D} = 40,123.50/25 = -1,604.94$

$S_D = \sqrt{24,254,458/24} = 1,005.29$

$S_{\bar{D}} = 1,005.29/\sqrt{25} = 156.01$

$$t = \frac{-1,604.94 - 0}{156.01} = -7.98$$

$t\ (.95;\ 24) = -1.711$

Reject Hypothesis 5

Table 13

Pairwise Test of Difference Between Expected Total Costs: $D = ETC(IP) - ETC(t_p,S1)$

Problem	D	$D-\overline{D}$	$(D-\overline{D})^2$
1	-294.97	-184.44	34,018
2	-191.52	- 80.99	6,559
3	-355.49	-244.96	60,005
4	-224.07	-113.54	12,891
5	-116.59	- 6.06	37
6	-271.47	-160.94	25,902
7	-124.30	- 13.77	190
8	12.02	122.55	15,018
9	- 66.14	44.39	1,970
10	160.62	271.15	73,522
11	-280.76	-170.23	28,978
12	-331.39	-220.86	48,779
13	152.00	262.53	68,922
14	-297.47	-186.94	34,947
15	-524.80	-414.27	171,620
16	83.99	194.52	37,838
17	33.32	143.85	20,693
18	580.16	690.69	477,053
19	-163.94	- 53.41	2,853
20	-298.78	-188.25	35,438
21	-233.13	-122.60	15,031
22	- 15.99	94.54	8,938
23	- 74.89	35.64	1,270
24	- 89.17	21.36	456
25	169.50	280.03	78,417
Total	-2,763.26		1,261,345

$$\overline{D} = -2,763.26/25 = -110.53$$

$$S_D = \sqrt{1,261,345/24} = 229.25$$

$$S_{\overline{D}} = 229.25/\sqrt{25} = 45.85$$

$$t = \frac{-110.53 - 0}{45.85} = -2.41$$

$$t\ (.95;\ 24) = -1.711$$

Reject Hypothesis 6

Table 14

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(t_p,S2)

Problem	D	D-$\bar{D}$	(D-$\bar{D}$)^2
1	-1,474.29	-579.53	335,855
2	-1,325.35	-430.59	185,408
3	-1,599.07	-704.31	496,053
4	-1,094.66	-199.90	39,960
5	-1,157.55	-262.79	69,059
6	- 792.15	102.61	10,529
7	-1,149.10	-254.34	64,689
8	- 471.72	423.04	178,963
9	-1,352.24	-457.48	209,288
10	- 265.04	629.72	396,547
11	-1,336.84	-442.08	195,435
12	-1,234.29	-339.53	115,281
13	-1,006.32	-111.56	12,446
14	- 990.93	- 96.17	9,249
15	-1,296.31	-401.55	161,242
16	- 724.64	170.12	28,941
17	- 444.51	450.25	202,725
18	- 169.48	725.28	526,031
19	- 782.91	111.85	12,510
20	-1,093.84	-199.08	39,633
21	- 553.94	340.82	116,158
22	- 631.03	263.73	69,554
23	- 617.37	277.39	76,945
24	- 522.12	372.64	138,861
25	- 283.19	611.57	374,018
Total	-22,368.89		4,065,376

$\bar{D}$ = -22,368.89/25 = -894.76

$S_D = \sqrt{4,065,376/24} = 411.57$

$S_{\bar{D}} = 411.57/\sqrt{25} = 82.31$

$t = \dfrac{-894.76 - 0}{82-31} = -10.87$

t (.95;24) = -1.711

Reject Hypothesis 7

Table 15

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(s,S1)

Problem	D	$D-\bar{D}$	$(D-\bar{D})^2$
1	-521.12	-776.62	603,139
2	-139.79	-395.29	156,254
3	295.74	40.24	1,619
4	- 77.03	-332.53	110,576
5	1.12	-254.38	64,709
6	745.33	489.63	239,933
7	-192.44	-447.94	200,650
8	409.80	154.30	23,808
9	120.85	-134.65	18,131
10	568.55	313.05	98,000
11	327.93	72.43	5,246
12	272.29	16.79	282
13	162.59	- 93.00	8,649
14	834.23	578.73	334,928
15	354.47	98.97	9,795
16	533.17	277.67	77,101
17	393.23	137.73	18,970
18	722.80	467.30	218,369
19	24.73	-230.77	53,255
20	- 34.81	-290.31	84,280
21	-164.61	-420.11	176,492
22	469.56	214.06	45,822
23	415.28	159.78	25,530
24	77.45	-178.05	31,702
25	788.39	532.89	283,972
Total	6,387.62		2,891,212

$$\bar{D} = 6,387.62/25 = 255.50$$

$$S_D = \sqrt{2,891,212/24} = 347.08$$

$$S_{\bar{D}} = 347.08/\sqrt{25} = 69.42$$

$$t = \frac{255.50 - 0}{69.42} = 3.68$$

$$t(.95;24) = .1.711$$

Cannot reject Hypothesis 8

Table 16

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(s,S2)

Problem	D	$D-\bar{D}$	$(D-\bar{D})^2$
1	-174.84	-527.87	278,647
2	345.76	- 7.27	53
3	319.60	- 33.43	1,118
4	113.45	-239.58	57,399
5	- 49.69	-402.72	162,183
6	801.54	448.51	201,161
7	605.26	252.23	63,620
8	534.04	181.01	32,765
9	131.59	-221.44	49,036
10	488.38	135.35	18,320
11	181.24	-171.79	29,512
12	365.62	12.39	154
13	394.76	41.73	1,741
14	918.02	564.99	319,214
15	365.56	12.53	157
16	700.06	347.03	120,430
17	299.53	- 53.50	2,862
18	834.44	481.41	231,756
19	99.98	-253.05	64,034
20	23.54	-329.49	108,564
21	- 36.20	-389.23	151,500
22	490.01	136.98	18,764
23	334.89	- 18.14	329
24	- 34.69	-387.72	150,327
25	773.86	420.83	177,098
Total	8,825.71		2,240,740

$\bar{D} = 8,825.71/25 = 353.03$

$S_D = \sqrt{2,240,740/24} = 305.56$

$S_{\bar{D}} = 305.56/\sqrt{25} = 61.11$

$t = \dfrac{353.03 - 0}{61.11} = 5.78$

$t\ (.95;24) = -1.711$

Cannot reject Hypothesis 9

postulated by the experiment. The (s,S) model differs from the mixed IP, EOQ, and (t_p,S) models in that it is not a periodic reorder model. Instead, the (s,S) model is a perpetual review model in which orders are placed whenever the stock level falls to some predetermined minimum level, s. This feature of the model, while resulting in higher stock monitoring and bookkeeping costs, permits the maintenance of a lower average stock level, and thus results in lower holding costs. The feature also permits a more immediate response during periods in which demand is considerably greater than predicted.

An additional test was conducted in order to determine whether the attractive perpetual review feature of the (s,S) model could be offset by decreasing the time between reorder points for the mixed-integer programming model. Problem 18, in which the (s,S) model performed considerably better than the mixed IP model during the original experiment, was reformulated as a 40-period problem. This reformulation was accomplished by halving estimated and realized periodic demands and the periodic holding cost. Stockout cost and reorder cost, which were assumed to be time-independent, remained unchanged.

The branch-and-bound integer programming algorithm was then used to determine mixed IP first-period reorder policies for 40 consecutive eight-period planning horizons. A computer code was next employed to compute the total costs

resulting from using the mixed IP model and both versions of the (s,S) model. These total costs are presented in Table 17.

The resultant total costs provided evidence to support the contention that the performance of the mixed IP model compares more favorably to that of the (s,S) model when the time between reorder points is shortened. Whereas the use of the mixed IP model had resulted in a cost that was 26.25 percent greater than the cost incurred by using the first version of the (s,S) model when 20 reorder points were considered, this difference dropped to 6.23 percent when 40 reorder points were considered. Similarly, the cost differential between the mixed IP model and the second version of the (s,S) model was reduced from 31.5 percent to 7.80 percent when the number of replenishment opportunities for the mixed IP model was increased from 20 to 40. The fact that the total cost incurred as a result of employing the mixed IP model was reduced by 21.44 percent when the number of reorder points was increased to 40 should also be noted. These results suggest that it may be possible to determine an optimal number of periods into which a finite planning horizon can be partitioned in order to implement the mixed IP model most effectively.

<u>Test for sensitivity to cost ratio</u>. The study was further concerned with ascertaining whether the performance of the mixed-integer programming model, relative to those of the

Table 17

Total Cost Comparisons, Mixed IP Model vs. (s,S) Model, Problem 18

Model	Total Cost: Four-Period Planning Horizon (20 periods)	Total Cost: Eight-Period Planning Horizon (40 periods)
Mixed IP	3473.84	2731.28
(s,S1)	2754.04	2591.62
(s,S2)	2642.40	2553.99

EOQ, (t_p,S), and (s,S) models is sensitive to the ratio of stockout cost to holding cost. Accordingly, six linear regressions were conducted to determine if the difference between ETC(IP) and the expected total cost resulting from employing each version of the other three models is significantly dependent upon the value of the cost ratio. The results of the regression analysis are summarized in Table 18.

The results of regressions 1 and 2 revealed the existence of significant linear relationships between the cost ratio and both [ETC(IP) - ETC(EOQ1)] and [ETC(IP) - ETC(EOQ2)]. The contention that there is no significant linear relationship between expected total cost differences and the cost ratio could not be rejected for any of the four remaining pairs of models.

The negative coefficients of regression that were computed in regressions 1 and 2 imply that, in both cases, the difference between expected total costs decreases in absolute magnitude as the ratio of stockout cost to holding cost becomes smaller. This observation raised the question of whether the mixed-integer programming model would continue to perform better than both versions of the EOQ model when stockout cost is less than holding cost. Nine of the 25 inventory problems that were generated for purposes of evaluating comparative model performance have ratios of stockout cost to holding cost that are less than 1. These nine problems were used to conduct two additional tests of

Table 18

Results of Bivariate Linear Regressions: Independent Variable X = Stockout Cost/Holding Cost

Regression Number and Dependent Variable	ß	Std. Error ß	R^2	F
1. ETC(IP) - ETC(EOQ1) ..	- 990.03	222.69	.4622	19.77**
2. ETC(IP) - ETC(EOQ2) ..	-1413.45	257.59	.5669	30.11**
3. ETC(IP) - ETC(t_p,S1) .	- 129.32	85.06	.0913	2.31**
4. ETC(IP) - ETC(t_p,S2) .	- 284.65	148.83	.1372	3.66**
5. ETC(IP) - ETC(s,S1) ..	137.35	131.99	.0450	1.08**
6. ETC(IP) - ETC(s,S2) ..	216.44	110.05	.1440	3.87**

difference between expected total costs. Pairwise Student's *t* tests were conducted in an attempt to reject the following null hypotheses:

<u>Hypothesis 4a</u>: ETC(IP) $\geq$ ETC(EOQ1)

<u>Hypothesis 5a</u>: ETC(IP) $\geq$ ETC(EOQ2)

The results of the pairwise *t* tests are presented in Tables 19 and 20. Both null hypotheses were rejected at the .05 level of significance. These test results led to the conclusion that, given the conditions postulated in the experiment, the mixed IP model performs better than both versions of the EOQ model when the ratio of stockout cost to holding cost is less than 1.

Summary

This chapter has presented the results of analyses conducted to evaluate the relative performances of the mixed IP model, the EOQ model, the (t_p,S) model, and the deterministic (s,S) model, when each was employed in the solution of a multiperiod inventory problem characterized by beta-distributed demands. The evaluation was accomplished through an experiment in which each model was used to solve a sample of 25 randomly generated inventory problems. Each problem required that replenishment decisions be made during 20 consecutive simulated periods. Replenishment decisions were based on cost and demand estimates that were available over a four-period finite planning horizon.

Table 19

Pairwise Test of Difference Between Expected Total Costs:
D = ETC(IP) - ETC(EOQ1) (Stockout Cost Less than Holding Cost)

Prob-lem	TC(IP)	TC(EOQ1)	D	$D-\bar{D}$	$(D-\bar{D})^2$
1	3869.53	4140.00	- 270.47	340.88	116,199
2	3476.84	4121.40	- 644.56	- 33.21	1,103
3	3009.72	3929.80	- 920.08	-308.73	95,314
4	4015.92	5246.40	-1230.48	-619.13	383,322
5	3743.35	4094.00	- 350.64	206.70	67,964
6	3436.40	393[illegible].[illegible]0	- 499.60	111.75	12,488
7	2899.19	4[illegible].20	-1264.01	-652.66	425,965
8	3725.41	4019.00	- 293.59	317.76	100,971
9	2528.89	2557.60	- 28.71	582.64	339,469
Total			-5502.15		1,542,795

$$\bar{D} = -5502.15/9 = -611.35$$

$$S_D = \sqrt{1,542,795/8} = 439.15$$

$$S_{\bar{D}} = 439.15/\sqrt{9} = 146.38$$

$$t = \frac{-611.35 - 0}{146.38} = -4.18$$

$$t\,(.95;8) = -1.86$$

Reject Hypothesis 4a

Table 20

Pairwise Test of Difference Between Expected Total Costs: D = ETC(IP) - ETC(EOQ2) (Stockout Cost Less than Holding Cost)

Problem	TC(IP)	TC(EOQ2)	D	$D-\bar{D}$	$(D-\bar{D})^2$
1	3869.53	4061.80	- 192.27	491.97	242,033
2	3476.84	4214.20	- 737.36	- 53.12	2,822
3	3009.72	3820.80	- 811.08	-126.84	16,089
4	4015.92	5556.80	-1540.88	-856.64	733,832
5	3743.35	4284.40	- 541.05	143.19	20,503
6	3436.40	4062.60	- 626.20	58.04	3,369
7	2899.19	4131.40	-1232.21	-547.97	300,271
8	3725.41	4054.80	- 329.39	354.85	125,919
9	2528.89	2676.00	- 147.71	536.53	287,864
Total			-6158.15		1,732,702

$\bar{D} = -6158.15/9 = 684.24$

$S_D = \sqrt{1,732,702/8} = 465.39$

$S_{\bar{D}} = 465.39/\sqrt{9} = 155.13$

$t = \dfrac{-684.24 - 0}{155.13} = 4.41$

$t\ (.95;8) = -1.86$

Reject Hypothesis 5a

The mixed IP model used expectations as demand inputs to obtain reorder policies. Each of the other three models used two different approaches in determining proxies for the constant demands assumed by the models. In the first approach, estimates of the most likely next-period demands were used as demand inputs. In the second approach, demand inputs consisted of averages of demand estimates over the next four periods.

An analysis of variance was conducted to test the hypothesis that, given the conditions postulated in the experiment, expected total cost is independent of the inventory model employed. Following rejection of this hypothesis, follow-up tests were conducted to evaluate the performance of the mixed IP model relative to the performance of each version of the other three models. The mixed IP model was found to perform significantly better, under the postulated conditions, than each version of both the EOQ and (t_p, S) models.

The performance of the mixed IP model was not found to be better than either version of the (s,S) model. Subsequent analysis demonstrated, however, that the performance of the mixed IP model, relative to that of the (s,S) model, could be improved substantially by increasing the number of opportunities for replenishment for the mixed IP model.

The sensitivity of expected total cost differences to the ratio of stockout cost to holding cost was assessed through simple linear regression analysis. Two expected

total cost differences, involving both versions of the EOQ model, were found to be sensitive to the cost ratio. Additional follow-up tests disclosed that the mixed IP model performs better than both versions of the EOQ model whether the ratio of stockout cost to holding cost is greater than or less than 1.

CHAPTER VI

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS FOR ADDITIONAL RESEARCH

Introduction

This research was concerned with the identification of inventory control methods for solving multiperiod inventory problems that are more representative of the environment in which much inventory control is practiced than are the conditions postulated by common inventory models. A review of the literature has revealed that, to date, only a very few special cases of stochastic, multiperiod, constrained inventory models have been solved, and then only after considerable analytical and computational effort. This study was concerned with the essential question of how to address inventory control when confronted with not unrealistic conditions for which modeling and solution procedures have not been developed.

Each of the pragmatic strategies considered in this research entailed the use of deterministic inventory models. In each instance, random variables were replaced by deterministic proxies to provide (nonoptimal) policies for implementation over a multiperiod planning horizon. These policies were implemented only during the immediate period. Expectations and policies were revised at the end of the

period in response to conditions that were realized during the period.

In addition to evaluating the relative performance of several pragmatic strategies, the research placed special emphasis on the best means of using deterministic mixed-integer programming models as proxies for stochastic multi-period inventory models. The special concern for the mixed-integer programming model resulted from recognition of the fact that, among all of the easy-to-use deterministic inventory models, the mixed IP formulation is exceptionally amenable to the additional constraints and multiple-objective criteria that coincide with broadly conceived statements of inventory control.

Summary and Conclusions

One major objective of the research was to determine whether total cost over a four-period planning horizon is sensitive to the reorder policy that is implemented for the first period. This question was answered by computing expected total costs using various reorder policies for period 1 and optimal policies (contingent on the first-period policy) for the remaining three periods. These computations, conducted over a broad sample of cost and beta-distributed demand patterns, permitted plotting expected total cost over the planning horizon as a function of the reorder quantity specified by the first-period policy. The curves obtained in this manner resulted in the rejection of

the hypothesis that there is no unique minimum expected total cost over a four-period planning horizon. Rejection of the hypothesis implied the existence of an optimal (and identifiable) first-period policy.

A second objective of the study was to determine whether an optimal first-period reorder policy can be easily and economically determined for a four-period, finite horizon inventory problem characterized by changing costs and demands. In pursuit of this objective, a subjective analysis was conducted of the programming and computer-processing time requirements attendant to the identification of an optimal first-period reorder policy. The computer programming modifications necessary to make the basic mixed IP computer algorithm more amenable to the multiperiod inventory problem were not found to require an excessive amount of programming expertise. A study of the computer-processing time required to solve a four-period inventory problem resulted in the conclusion that, by using a combination of simulation and mixed-integer programming methods, an optimal first-period policy can be determined over a four-period planning horizon with a nominal investment in computer processing time.

A third major objective of the research was to ascertain the significance of the error that is introduced by applying a mixed-integer programming formulation as a deterministic proxy to a probabilistic inventory problem. This objective was pursued by using analysis of variance techniques to assess the adequacy of first-period reorder

policies obtained by using the mixed IP model when expectations are used as periodic demands over a four-period planning horizon. An analysis of variance, conducted at the .01 significance level, revealed that the expected total cost incurred by implementing mixed IP first-period policies could not be shown to differ significantly from the expected total cost resulting from the implementation of optimal first-period policies. This disclosure led to the conclusion that first-period policies obtained by using the mixed IP model with expectations as periodic demand inputs are generally adequate under the conditions specified in the research.

The fourth major objective addressed by this research was concerned with evaluating the performance of the mixed-integer programming formulation relative to the performances of the EOQ model, the (t_p,S) model, and the deterministic (s,S) model, when each is employed to solve a multiperiod problem characterized by beta-distributed demands. To accomplish this objective, an experiment was designed and conducted in which each model was used to solve a sample of 25 randomly generated inventory problems. Each problem required that replenishment decisions be made during 20 consecutive simulated periods. Replenishment decisions were based on available cost and demand estimates over a four-period finite planning horizon.

The mixed-integer programming formulation used expectations as periodic demands to obtain replenishment policies. Each of the other three models used two different

approaches in establishing proxies for the constant demands assumed by the models. In the first approach, periodic demand inputs were based on estimates of the most likely next-period demands. In the second approach, demand inputs were determined by averaging demand estimates over the next four periods.

The hypothesis that expected total cost is independent of the model employed, given the conditions postulated in the experiment, was rejected using analysis of variance techniques. Follow-up tests were then conducted to evaluate the performance of the mixed-integer programming formulation relative to the performance of each version of the other three models. In these tests, the mixed IP model was found to perform significantly better, given the experimental conditions, than each version of both the EOQ and (t_p,S) models.

The performance of the mixed IP model was not found to be superior to either version of the (s,S) model. This result was not unexpected, however, given the nature of the (s,S) model and the limited conditions postulated in the experiment. The perpetual review feature of the (s,S) model permits the maintenance of a lower average stock level, and thus results in lower holding costs. This feature, coupled with the assumption of instantaneous replenishment, also permits a more immediate response during periods in which actual demand is considerably greater than predicted. During the design of the experiment, no conscious effort was made

to establish inventory costs at levels that would have tended to offset these apparent advantages of the (s,S) model. Instead, the various inventory costs were randomly determined over ranges that were consistent with earlier portions of this study. In addition, the experiment was not designed to reflect the higher stock monitoring and bookkeeping costs that are attendant to a perpetual review model such as the (s,S) model.

An additional test was conducted to determine whether decreasing the time between reorder points for the mixed-integer programming model could offset the attractive perpetual review feature of the (s,S) model. A problem in which the (s,S) model had originally performed considerably better than the mixed IP model was reformulated as a 40-period problem. A computer code was then used to compute the total costs resulting from using the mixed IP model and both versions of the (s,S) model to solve the problem. The reductions in the cost differentials between the mixed IP model and each version of the (s,S) model were significant. These reductions provided evidence to support the contention that the performance of the mixed IP model, relative to the performance of the (s,S) model, could be improved substantially by increasing the number of opportunities for replenishment for the mixed IP model.

A further concern of the study was to ascertain whether the performance of the mixed-integer programming model, relative to those of the EOQ, (t_p,S), and (s,S)

models is sensitive to the ratio of stockout cost to holding cost. Accordingly, linear regression analysis was used to assess the sensitivity of expected total cost differences to the cost ratio. The analysis disclosed that two expected total cost differences, involving both versions of the EOQ model, were sensitive to the cost ratio. Additional follow-up tests, however, permitted the conclusion that the mixed IP model performs better, given the experimental conditions, than both versions of the EOQ model whether the ratio of stockout cost to holding cost is greater than or less than 1.

In this study, the researcher manually accomplished the data processing procedures, including the modification of an available branch-and-bound computer algorithm, required to facilitate the identification of optimal first-period inventory policies. With comparatively little additional effort, the basic computer code can be further modified to make the approach more accessible to practitioners. These additional modifications would require only that the practitioner provide 3-parameter estimates of periodic demands. It is believed that such estimates can be economically obtained through an analysis of corporate records pertaining to sales forecasts, production capabilities, etc. The cost of subsequent optimization has been shown to be nominal. These factors, coupled with the increased availability of computer algorithms and the current wide-spread accessibility of data processing equipment to companies of all sizes, suggest that the continued extensive

use of simple lot-size models is highly questionable. Considerably better results can be achieved by using the more powerful mixed-integer programming inventory model that was described in this study.

As a result of one of the findings of this research, there is strong evidence that the number of periodic reorder points required to make the mixed IP model equivalent in performance to the deterministic (s,S) model is reasonably small. When the model is embellished with additional constraints, this conclusion may not continue to hold. However, as the mixed IP model is embellished to more closely reflect the realities of the inventory problem being modeled, its applications-oriented superiority over the (s,S) model appears to be highly probable.

The most persistent criticism of multistage stochastic inventory models has been that the computational and tractability burdens imposed by these models make them of little or no value to the practitioner. In this study, very adequate results were obtained by using a deterministic mixed IP model as a proxy for the stochastic inventory model. These results tend to support the contention of others that efforts to develop complex and unwieldy stochastic optimization models might be more profitably directed toward explorations of approximation methods.

The multiperiod inventory problem addressed in this research is recognized as a special case of the general multistage stochastic programming problem. Reflection on the techniques and methodology that were employed in the study suggests that it may be possible to extend these techniques and the methodology to other optimization problems, such as capital budgeting and queueing problems.

The basic research methodology - namely, simulation and optimization techniques - is by no means unique to problems characterized by beta-distributed periodic demands. Although this study was limited to the consideration of only beta-distributed demands, there is no reason to believe that the basic methodology would be any less appropriate if periodic demands were known to be distributed according to other probability distributions.

The data requirements for using the basic procedures described in this research coincide very closely with the information structure of material requirements planning (MRP) systems. Hence, the procedures for identifying optimal periodic reorder quantities can be easily coupled with existing MRP systems. It would appear evident that, in many circumstances, the inclusion with an MRP system of an inventory control model of the form described in this study would serve to enhance inventory decision making.

Recommendations for Additional Research

This study was necessarily limited in scope. During the course of the research, however, several opportunities for appropriate follow-on studies became apparent. Some suggestions for additional research in the area follow.

Coupling the mixed IP model and MRP systems. One potential follow-on study would provide a discussion of the manner in which a mixed-integer programming inventory model, such as the model developed in this research, may be embedded within a material requirements planning (MRP) system. An MRP system is an information system that is designed to translate an organization's master production schedule into time-phased net requirements, and the planned coverage of such requirements, for each component inventory item needed to implement the schedule. MRP has become an increasingly popular method of controlling inventories. Proponents of MRP systems have resisted the use of embedded inventory models primarily because of the unrealistic assumptions attendant to these models.

The mixed IP inventory model is devoid of many of the assumptions that are bothersome to the advocates of MRP. The model can be tailored to reflect the unique specifications and restrictions of the inventory system being modeled. Such a model could not only provide day-to-day assistance in inventory management, but could also be used by management to assess the overall effects of alternative managerial decisions. This assessment could be accomplished by varying

specific parameters within the information system and using the model to compute resultant expected costs.

A study in which a mixed IP model is embedded within an existing MRP system would be of particular interest. Such a study would serve to ascertain the feasibility of such an approach to the inventory control problem, and would also provide a guide that interested practitioners might follow.

Demand distributions. This research was limited to consideration of multi-period inventory problems in which periodic demands are assumed to follow a generalized-beta distribution. A worthwhile follow-on effort would entail an evaluation of the applicability of the general solution procedures advanced in this study when periodic demands are known to follow a probability distribution other than a beta distribution. Such a research effort might also include a discussion of the magnitude of the error introduced by assuming beta-distributed periodic demands when actual periodic demands are better described by another distribution. Of particular interest would be an assessment of possible errors when extreme demand distributions, such as the uniform and quasi-delta distributions, are encountered.

A relatively short study could also be devoted to a discussion of the errors introduced as a result of using approximations for the mean and standard deviation of the beta-distributed demand variable, rather than the exact values for these parameters. Such a discussion might lead to

the suggestion that a probability distribution other than the generalized-Beta distribution may be more generally applicable to problems in which the actual demand distribution is unknown.

Extension of the basic mixed IP inventory model. The mixed-integer programming model that was formulated in this research is applicable to only a comparatively simple inventory problem. An additional research effort might present a discussion of how the basic IP model can be modified to incorporate a host of additional, pervasive considerations.

Such a study might demonstrate, for example, how the objective function of the model can be easily modified to accommodate additional managerial objectives such as the minimization of the present worth of (future) total inventory costs, or the minimization of tax expense. The study could show how, by introducing additional constraints, the basic model can be extended to consider factors such as storage capacity restrictions, cash flow limitations, and maximum replenishment restrictions.

The study could also illustrate how the basic model can be modified to accommodate multiple inventory items and multiple vendors. A discussion could also be provided of the manner in which the model can be extended to permit replenishment stock to arrive throughout a period, rather than only at the beginning of a period. A similar discussion could show how the model can be revised to allow back orders to be filled.

A study in which the researcher formulates a mixed IP model to fit an existing inventory system would be especially beneficial. Such a study should emphasize the ease with which a practitioner can interact with the model.

<u>Optimal reorder points</u>. During the course of this research, an analysis of the relative performance of the mixed IP model revealed that, under the conditions postulated in the experiment, expected total cost over the planning horizon decreased as the number of opportunities for replenishment were increased. A follow-on study in which a method might be developed for determining the optimal number of replenishment opportunities over a given planning horizon would be particularly beneficial.

<u>Additional tests of comparative model performance</u>. In this research, the evaluation of comparative model performance was limited to consider, in addition to the mixed-integer programming model, three well-known deterministic inventory models - specifically, the EOQ model, the (t_p,S) model, and the (s,S) model. An additional study could be conducted to contrast the performance of the mixed IP model with the performances of other commonly used models. A very challenging research effort would involve comparing the performance of the mixed IP model to that of the multi-period stochastic (s,S) model.

Variable costs of inventory. In this study, it was concluded that, given the postulated experimental conditions, first-period reorder policies obtained by using the mixed IP model with expectations as periodic demand inputs are generally adequate. Additional research should be conducted to determine if a similar approach may be applied when confronted with problems characterized by stochastic inventory costs.

Relation to the general problem. A particularly valuable follow-on study would relate the multiperiod inventory control procedures employed in this research to the general multistage stochastic programming problem. Such a study would serve to put the findings of this research in the perspective of a more general problem.

The multiperiod inventory problem discussed in this research may be identified as being a special case of the general multistage stochastic programming problem in which the decision maker has recourse to variable policies after the first-period solution has been obtained. Suggested solution procedures for the general problem have been based primarily on attempts to include constraints for all possible realizations of random variables. In the presence of continuous random variables, the model becomes unbounded with respect to the order of the constraint matrix. No practical solution has yet been developed for solving this classical operations research problem.

In research conducted preliminary to this dissertation, the researcher attempted to obtain an approximate solution to the problem through the combined usage of mixed-integer goal programming and computer simulation techniques. The intent was to develop a generally applicable decomposition method that would require modeling only those conditions that would have to be satisfied given one vector of realizations for random variables. Starting with the optimal solution for one realization, the intention was to then sequentially apply goal programming methods to subsequent realizations in a manner that would penalize new solutions that failed to satisfy previously examined realizations. In many respects, the intended solution technique could be regarded as being a search method that is based on sequential sampling and sensitivity analysis. The well-known difficulties encountered in using sensitivity analysis in <u>integer</u> programming applications led to the abandonment of the approach in this research. Nonetheless, the approach might prove to be a workable method for stochastic extensions of linear programming models.

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